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$l \in \mathbb{Z}_{>0}$, $\lambda \in (\bar{P}^+)_l$, $\lambda(c) = l$, $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$
 $\mathfrak{g} =$ symmetrizable affine Lie algebra

Classical crystal:

A crystal B for $U'_q(\mathfrak{g})$ -module V is called a classical crystal.

Perfect Crystal of level l :

A classical crystal B is a perfect crystal of level l if the following hold:

- 1) \exists a finite dim'l $U'_q(\mathfrak{g})$ -module V with crystal B .
- 2) $B \otimes B$ is connected
- 3) $\exists \lambda_0 \in \bar{P}$ such that $\#B_{\lambda_0} = 1$ and $\overline{\text{wt}}(B) \subset \lambda_0 + \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i$.
- 4) $\langle c, \varepsilon(b) \rangle \geq l \quad \forall b \in B$
- 5) For each $\lambda \in \bar{P}^+_l \exists!$ vectors $b^\lambda, b_\lambda \in B$ such that $\varepsilon(b^\lambda) = \lambda$, $\varphi(b_\lambda) = \lambda$.

Set $B^{\min} = \{b \in B \mid \langle c, \varepsilon(b) \rangle = l\}$
 = set of minimal elements
 in B .

5) $\Rightarrow \varepsilon, \varphi : B^{\min} \longrightarrow \bar{P}_l^+$ bijections.

Ex(1)

$$U_q(\mathfrak{g}), \quad \mathfrak{g} = \widehat{sl}(3, \mathbb{C})$$

$$\bar{\mathfrak{g}} = sl(3, \mathbb{C})$$

Consider the $\bar{\mathfrak{g}}$ -module $V(\bar{\Lambda}_1) = F^3$,
 $F = \mathbb{C}(q)$

Crystal for $V(\bar{\Lambda}_1)$ is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

We can define a $U_q(\mathfrak{g})$ action on $V(\bar{\Lambda}_1)$
 with the 0-action on the crystal
 given by

$$\boxed{1} \xleftarrow{0} \boxed{2} \xrightarrow{2} \boxed{3} : B$$

$$\bar{P}_1^+ = \{\Lambda_0, \Lambda_1, \Lambda_2\}$$

$$\varepsilon(\boxed{1}) = \Lambda_0$$

$$\varphi(\boxed{1}) = \Lambda_1$$

$$\varepsilon(\boxed{2}) = \Lambda_1$$

$$\varphi(\boxed{2}) = \Lambda_2$$

$$\varepsilon(\boxed{3}) = \Lambda_2$$

$$\varphi(\boxed{3}) = \Lambda_0$$

$$b^{\Lambda_0} = \boxed{1}, \quad b^{\Lambda_1} = \boxed{2}, \quad b^{\Lambda_2} = \boxed{3}$$

$$b_{\Lambda_0} = \boxed{3}, \quad b_{\Lambda_1} = \boxed{1}, \quad b_{\Lambda_2} = \boxed{2}$$

$$B^{\min} = \{ \boxed{1}, \boxed{2}, \boxed{3} \}$$

$B \otimes B$ connected (exer.)

$\therefore B$ is a perfect crystal of level 1, usually we denote $B = B^{1,1}$.

Ex(2) $\mathfrak{g} = C_2^{(1)} = \widehat{sp}(4, \mathbb{C})$, $\bar{\mathfrak{g}} = C_2 = sp(4, \mathbb{C})$

Consider the vector representation

$V(\bar{\Lambda}_1)$ of $\bar{\mathfrak{g}}$ whose crystal is:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

We can now define a $U_q'(\mathfrak{g})$ -action on $V(\bar{\Lambda}_1)$ whose crystal is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}} : B$$

This crystal B is not perfect since

we have $b^{\wedge 1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ not unique.

Ex(3) $\mathfrak{g} = C_2^{(1)}, \bar{\mathfrak{g}} = C_2$

$U_q(\bar{\mathfrak{g}})$ -module $V(\bar{\Lambda}_2)$ has crystal

$$\begin{array}{ccccccccc} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \xrightarrow{2} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \xrightarrow{1} & \begin{bmatrix} 2 \\ 2 \end{bmatrix} & \xrightarrow{1} & \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \xrightarrow{2} & \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ b_1 & & b_2 & & b_3 & & b_4 & & b_5 \end{array}$$

$V(\bar{\Lambda}_2)$ can be lifted to a $U_q'(C_2^{(1)})$ -module with crystal

$$\begin{array}{ccccccc} & & & \circ & & & \\ & \swarrow & & \searrow & & & \\ b_1 & \xrightarrow{2} & b_2 & \xrightarrow{1} & b_3 & \xrightarrow{1} & b_4 & \xrightarrow{2} & b_5 & : B \\ & & \nwarrow & & \nearrow & & & & & \\ & & & \circ & & & & & & \end{array}$$

$$(\bar{P}^+)_{cl} = \{\Lambda_0, \Lambda_1, \Lambda_2\}$$

$$b^{\wedge 0} = b_1$$

$$b_{\Lambda_0} = b_5$$

$$b^{\wedge 1} = b_3$$

$$b_{\Lambda_1} = b_3$$

$$b^{\wedge 2} = b_5$$

$$b_{\Lambda_2} = b_1$$

$B = \{b_1, b_3, b_5\}$, $B = B^{2,1}$ is a perfect crystal of level 1.

Let $\lambda \in \bar{P}_l^+$, $\bar{P} = \bigoplus_{i=0}^n \mathbb{Z}\Lambda_i$, $P = \bar{P} \oplus \mathbb{Z}\delta$
 $l \in \mathbb{Z}_{>0}$

\mathfrak{g} = affine Lie algebra

Suppose B_l be a perfect crystal of level l . Then

$\exists! b_\lambda \in B_l$ such that $\varphi(b_\lambda) = \lambda$

Thm: $B(\lambda) \cong B(\varepsilon(b_\lambda)) \otimes B_l$

as $U_q(\mathfrak{g})$ crystals. This isomorphism maps $u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$

Set $\lambda_1 = \lambda$, $b_1 = b_\lambda$ and define

$$\lambda_{k+1} = \varepsilon(b_{\lambda_k}), \text{ and } b_{k+1} = b_{\lambda_{k+1}}$$

for $k = 1, 2, 3, \dots$

Applying above thm repeatedly we get

$$B(\lambda) \cong B(\lambda_{k+1}) \otimes B_l^{\otimes k}$$

$$u_\lambda \mapsto u_{\lambda_{k+1}} \otimes b_k \otimes \dots \otimes b_2 \otimes b_1$$

Define

$$P_\lambda = \dots \otimes b_k \otimes b_{k-1} \otimes \dots \otimes b_2 \otimes b_1$$

P_λ is called the λ -ground state path in B_ℓ .

A λ -path in B_ℓ is

$$\dots \otimes p(k) \otimes p(k-1) \otimes \dots \otimes p(2) \otimes p(1)$$

with $p(j) \in B_\ell$ and $p(k) = b_k$ for $k \gg 0$

Denote

$P(\lambda, B_\ell)$ = set of all λ -paths in B_ℓ which is a $U'_q(\mathfrak{g})$ -crystal.

Thm: $B(\lambda) \cong P(\lambda, B_\ell)$

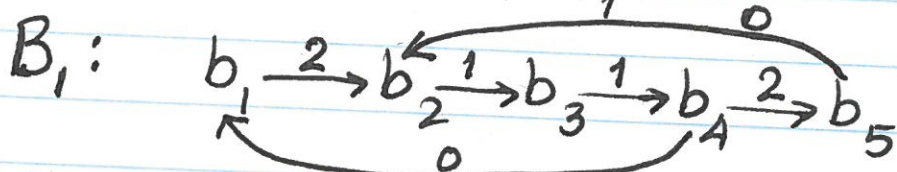
as $U'_q(\mathfrak{g})$ crystals. (This is called the path realization of the affine crystal $B(\lambda)$.)

Energy Function: For the perfect crystal B_ℓ , \exists a \mathbb{Z} -valued function:

$H: B_\ell \otimes B_\ell \rightarrow \mathbb{Z}$,
 called energy function, such that for $i \in I$, $b \otimes b' \in B_\ell \otimes B_\ell$, $\tilde{e}_i(b \otimes b') \neq 0$, we have

$$H(\tilde{e}_i(b \otimes b')) = \begin{cases} H(b \otimes b'), & i \neq 0 \\ H(b \otimes b') + 1, & i = 0, \varphi_0(b) \geq \varepsilon_0(b') \\ H(b \otimes b') - 1, & i = 0, \varphi_0(b) < \varepsilon_0(b') \end{cases}$$

Example: $\mathfrak{g} = C_2^{(1)}$, $B_1 = B_0^{2,1}$



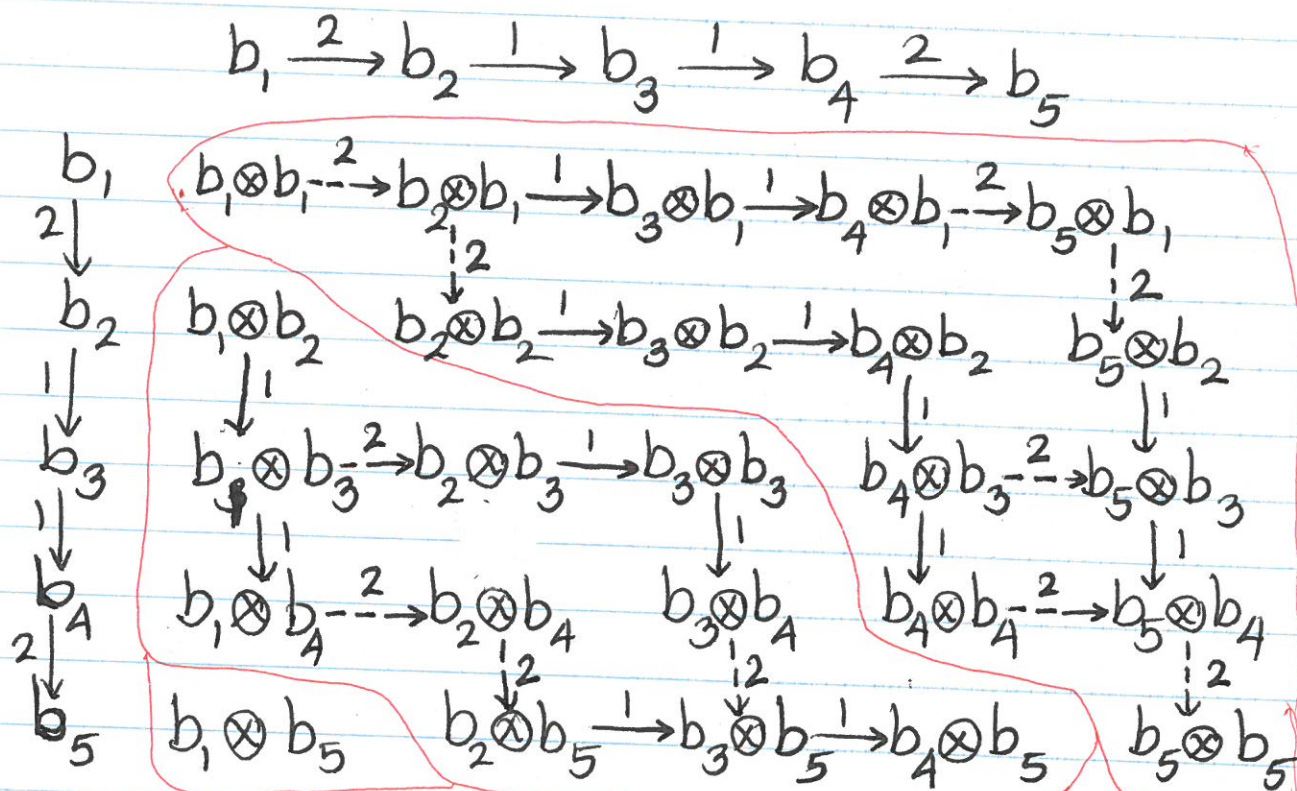
Note that, deleting the 0-arrows,

$B_1 \cong B(\bar{\Lambda}_2)$: $b_1 \xrightarrow{2} b_2 \xrightarrow{1} b_3 \xrightarrow{1} b_4 \xrightarrow{2} b_5$
 as C_2 crystal.

Let us decompose $B(\bar{\Lambda}_2) \otimes B(\bar{\Lambda}_2)$:

$$B(\bar{\Lambda}_2) \otimes B(\bar{\Lambda}_2) \cong B(2\bar{\Lambda}_2) \oplus B(2\bar{\Lambda}_1) \oplus B(0)$$

as shown in next page.



Three connected components with highest weight vectors:

$$b_1 \otimes b_1, \quad b_1 \otimes b_2, \quad \text{and} \quad b_1 \otimes b_5$$

of wts: $2\bar{\lambda}_2$ $2\bar{\lambda}_2 - \alpha_2 = 2\bar{\lambda}_1$ $2\bar{\lambda}_2 - 2\alpha_1 - 2\alpha_2 = 0$.

These three connected components are linked by 0-arrows. Since

$$\tilde{e}_0(b_1 \otimes b_5) = b_4 \otimes b_5 \quad \text{and} \quad \tilde{e}_0(b_1 \otimes b_1) = b_1 \otimes b_4$$

we conclude that

$$H(b \otimes b') = \begin{cases} 2 & \text{if } b \otimes b' \in B(2\bar{\lambda}_2) \\ 1 & \text{if } b \otimes b' \in B(2\bar{\lambda}_1) \\ 0 & \text{if } b \otimes b' \in B(0) \end{cases}$$

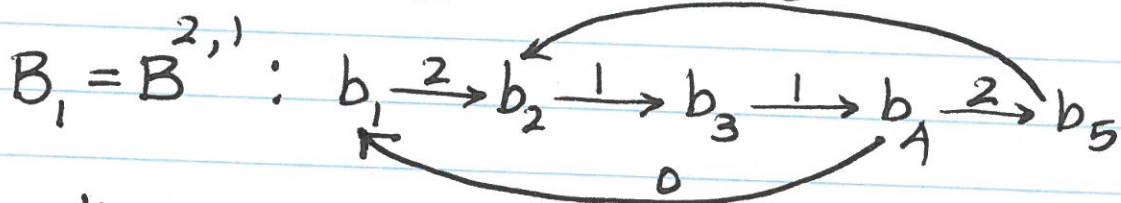
Using the energy function $H: B_\ell \otimes B_\ell \rightarrow \mathbb{Z}$
 the affine weight of a path

$$P = \dots \otimes p(k) \otimes \dots \otimes p(2) \otimes p(1) \in P(\lambda, B_\ell)$$

is given by the following formula:

$$\text{wt}(P) = \lambda + \sum_{k=1}^{\infty} (\text{wt}(p(k)) - \text{wt}(b_k)) - a_0^{-1} \delta \left(\sum_{k=1}^{\infty} k (H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k)) \right)$$

Ex: $\mathfrak{g} = C_2^{(1)}$



$$(P_{cl}^+)_1 = \{ \Lambda_0, \Lambda_1, \Lambda_2 \}$$

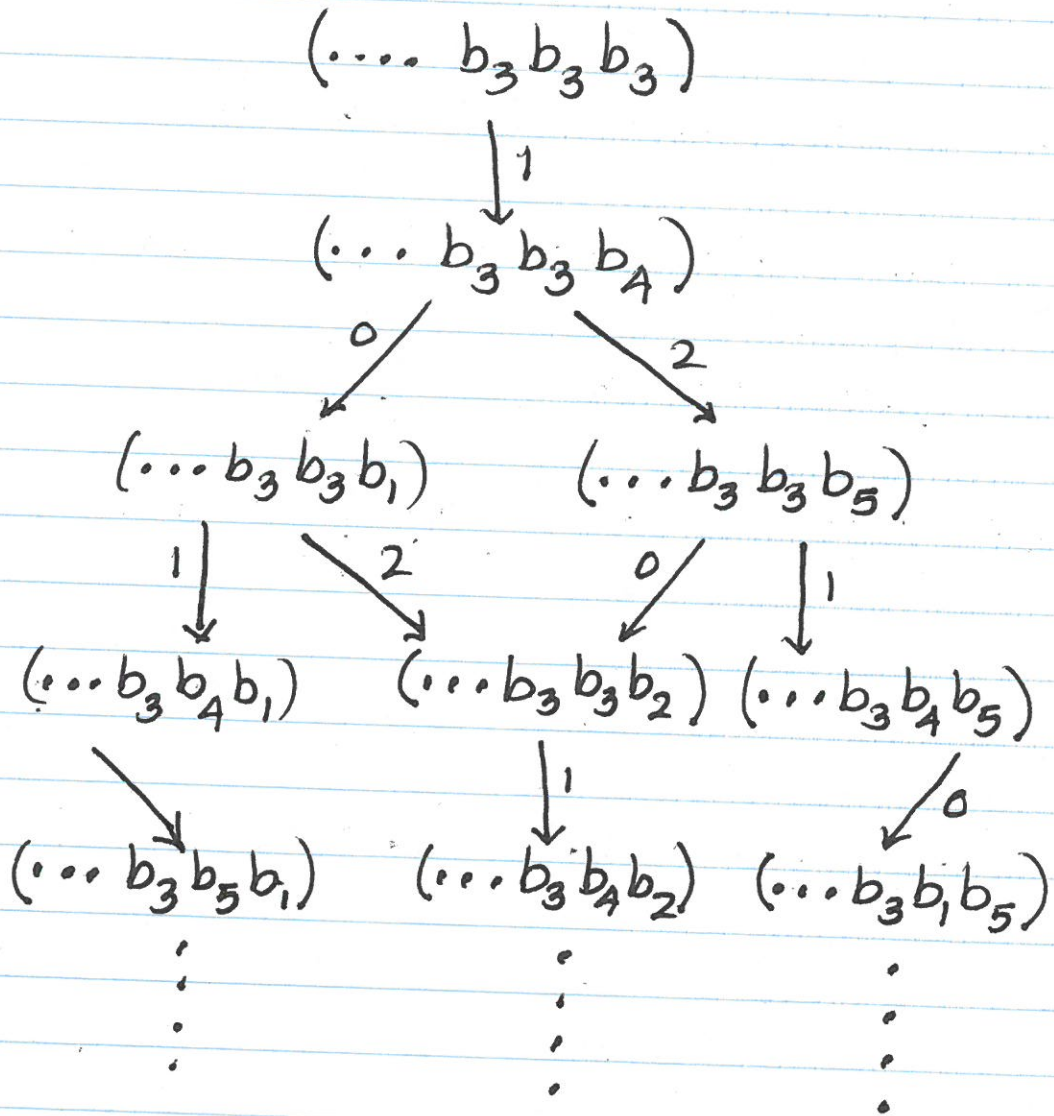
Λ_0 -ground state path: $\dots \otimes b_5 \otimes b_1 \otimes b_5$

Λ_1 -ground state path: $\dots \otimes b_3 \otimes b_3 \otimes b_3$

Λ_2 -ground state path: $\dots \otimes b_1 \otimes b_5 \otimes b_1$

Now using signature rule we can draw

the path realization of the affine crystal $B(\Lambda_1)$ as follows:



(Here we have dropped the tensor product symbol for convenience of presentation.)