4/26

$l \in \mathbb{Z}_{>0}$, $\lambda \in (\bar{P}^+_{cl})_l$, $\lambda(c) = l$, $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$

$\mathfrak{g}$ = symmetrizable affine Lie algebra

Classical crystal:

A crystal B for $U_q'(\mathfrak{g})$-module V is called a classical crystal.

Perfect Crystal of level l:

A classical crystal B is a perfect crystal of level l if the following hold:

1) $\exists$ a finite dim'l $U_q'(\mathfrak{g})$-module V with crystal B.

2) $B \otimes B$ is connected

3) $\exists\ \lambda_0 \in \bar{P}$ such that $\# B_{\lambda_0} = 1$ and $\overline{wt}(B) \subset \lambda_0 + \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i$.

4) $\langle c, \varepsilon(b) \rangle \geq l \ \forall\ b \in B$

5) For each $\lambda \in \bar{P}^+_l$ $\exists!$ vectors $b^\lambda, b_\lambda \in B$ such that $\varepsilon(b^\lambda) = \lambda$, $\varphi(b_\lambda) = \lambda$.

Set $B^{min} = \{ b \in B \mid \langle c, \varepsilon(b) \rangle = \ell \}$

$= $ set of minimal elements in B.

5) $\Rightarrow$ $\varepsilon, \varphi : B^{min} \longrightarrow \bar{P}^+_\ell$ bijections.

Ex(1)

$U_q(\mathfrak{g})$, $\mathfrak{g} = \widehat{s\ell}(3, \mathbb{C})$

$\bar{\mathfrak{g}} = s\ell(3, \mathbb{C})$

Consider the $\bar{\mathfrak{g}}$-module $V(\bar{\Lambda}_1) = F^3$,

$F = \mathbb{C}(q)$

Crystal for $V(\bar{\Lambda}_1)$ is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

We can define a $U_q'(\mathfrak{g})$ action on $V(\bar{\Lambda}_1)$ with the 0-action on the crystal given by

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[ll, bend right, "0"'] \end{tikzcd} \quad : B$$

$\bar{P}^+_1 = \{ \Lambda_0, \Lambda_1, \Lambda_2 \}$

$\varepsilon(\boxed{1}) = \Lambda_0$ $\qquad$ $\varphi(\boxed{1}) = \Lambda_1$

$\varepsilon(\boxed{2}) = \Lambda_1$ $\qquad$ $\varphi(\boxed{2}) = \Lambda_2$

$\varepsilon(\boxed{3}) = \Lambda_2$ $\qquad$ $\varphi(\boxed{3}) = \Lambda_0$

$b^{\Lambda_0} = \boxed{1}$, $b^{\Lambda_1} = \boxed{2}$, $b^{\Lambda_2} = \boxed{3}$

$b_{\Lambda_0} = \boxed{3}$, $b_{\Lambda_1} = \boxed{1}$, $b_{\Lambda_2} = \boxed{2}$

$B^{min} = \{ \boxed{1}, \boxed{2}, \boxed{3} \}$

$B \otimes B$ connected (exer.)

$\therefore$ B is a perfect crystal of level 1, usually we denote $B = B^{1,1}$.

<u>Ex(2)</u> $\mathfrak{g} = C_2^{(1)} = \widehat{sp}(4, \mathbb{C})$, $\bar{\mathfrak{g}} = C_2 = sp(4, \mathbb{C})$

Consider the vector representation $V(\bar{\Lambda}_1)$ of $\bar{\mathfrak{g}}$ whose crystal is:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

We can now define a $U_q'(\mathfrak{g})$-action on $V(\bar{\Lambda}_1)$ whose crystal is

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \arrow[lll, bend right, "0"'] \end{tikzcd} \; : B$$

This crystal B is <u>not</u> perfect since we have $b^{\Lambda_1} = \boxed{2}, \boxed{\bar{1}}$ <u>not</u> unique.

<u>Ex(3)</u> $\mathfrak{g} = C_2^{(1)}$, $\bar{\mathfrak{g}} = C_2$

$U_q(\bar{\mathfrak{g}})$-module $V(\bar{\Lambda}_2)$ has crystal

$$\begin{array}{ccccccccc} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} & \xrightarrow{2} & \begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline \end{array} & \xrightarrow{1} & \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline \end{array} & \xrightarrow{1} & \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline \end{array} & \xrightarrow{2} & \begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline \end{array} \\ \| & & \| & & \| & & \| & & \| \\ b_1 & & b_2 & & b_3 & & b_4 & & b_5 \end{array}$$

$V(\bar{\Lambda}_2)$ can be lifted to a $U_q'(C_2^{(1)})$-module with crystal

$$\begin{tikzcd} b_1 \arrow[r, "2"] & b_2 \arrow[r, "1"] & b_3 \arrow[r, "1"] & b_4 \arrow[r, "2"] \arrow[lll, bend right, "0"'] & b_5 \arrow[lll, bend left, "0"] \end{tikzcd} : B$$

$(\bar{P}_{cl}^+)_1 = \{\Lambda_0, \Lambda_1, \Lambda_2\}$

$b^{\Lambda_0} = b_1$ $\qquad b_{\Lambda_0} = b_5$

$b^{\Lambda_1} = b_3$ $\qquad b_{\Lambda_1} = b_3$

$b^{\Lambda_2} = b_5$ $\qquad b_{\Lambda_2} = b_1$

$B^{min} = \{b_1, b_3, b_5\}$, $B = B^{2,1}$ is a perfect crystal of level 1.

Let $\lambda \in \bar{P}^+_\ell$, $\bar{P} = \bigoplus\sum_{i=0}^{n} \mathbb{Z}\Lambda_i$, $P = \bar{P} \oplus \mathbb{Z}\delta$

$\ell \in \mathbb{Z}_{>0}$

$\mathfrak{g}$ = affine Lie algebra

Suppose B_ℓ be a perfect crystal of level ℓ. Then

$\exists!\ b_\lambda \in B_\ell$ such that $\varphi(b_\lambda) = \lambda$

<u>Thm</u>: $B(\lambda) \cong B(\varepsilon(b_\lambda)) \otimes B_\ell$

as $U_q'(\mathfrak{g})$ crystals. This isomorphism maps $u_\lambda \longmapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$

Set $\lambda_1 = \lambda$, $b_1 = b_\lambda$ and define

$$\lambda_{k+1} = \varepsilon(b_{\lambda_k}), \text{ and } b_{k+1} = b_{\lambda_{k+1}}$$

for $k = 1, 2, 3, \ldots$

Applying above thm repeatedly we get

$$B(\lambda) \cong B(\lambda_{k+1}) \otimes B_\ell^{\otimes k}$$

$$u_\lambda \longmapsto u_{\lambda_{k+1}} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1$$

Define
$$P_\lambda = \cdots \otimes b_k \otimes b_{k-1} \otimes \cdots \otimes b_2 \otimes b_1$$
P_λ is called the λ-ground state path in B_ℓ.

A a λ-path in B_ℓ is
$$\cdots \otimes p(k) \otimes p(k-1) \otimes \cdots \otimes p(2) \otimes p(1)$$
with $p(j) \in B_\ell$ and $p(k) = b_k$ for $k >> 0$

Denote
$$P(\lambda, B_\ell) = \text{set of all } \lambda\text{-paths in } B_\ell$$
which is a $U_q'(\mathfrak{g})$-crystal.

<u>Thm:</u> $$B(\lambda) \cong P(\lambda, B_\ell)$$
as $U_q'(\mathfrak{g})$ crystals. (This is called the path realization of the affine crystal $B(\lambda)$.

<u>Energy Function</u>: For the perfect crystal B_ℓ, $\exists$ a $\mathbb{Z}$-valued function:

$$H : B_\ell \otimes B_\ell \longrightarrow \mathbb{Z},$$

called energy function, such that for $i \in I$, $b \otimes b' \in B_\ell \otimes B_\ell$, $\tilde{e}_i(b \otimes b') \neq 0$, we have

$$H(\tilde{e}_i(b\otimes b')) = \begin{cases} H(b\otimes b'), & i \neq 0 \\ H(b\otimes b')+1, & i=0,\ \varphi_0(b) \geq \varepsilon_0(b') \\ H(b\otimes b')-1, & i=0,\ \varphi_0(b) < \varepsilon_0(b') \end{cases}$$

<u>Example</u>: $\mathfrak{g} = C_2^{(1)}$, $B_1 = B^{2,1}$

B_1:
$$\begin{tikzcd} b_1 \arrow[r, "2"] & b_2 \arrow[r, "1"] & b_3 \arrow[r, "1"] & b_4 \arrow[r, "2"] \arrow[lll, bend left, "0"] & b_5 \arrow[lll, bend right, "0"'] \end{tikzcd}$$

Note that, deleting the 0-arrows,

$$B_1 \cong B(\bar{\Lambda}_2): \ b_1 \xrightarrow{2} b_2 \xrightarrow{1} b_3 \xrightarrow{1} b_4 \xrightarrow{2} b_5$$

as C_2 crystal.

Let us decompose $B(\bar{\Lambda}_2) \otimes B(\bar{\Lambda}_2)$:

$$B(\bar{\Lambda}_2) \otimes B(\bar{\Lambda}_2) \cong B(2\bar{\Lambda}_2) \oplus B(2\bar{\Lambda}_1) \oplus B(0)$$

as shown in next page.

$$
\begin{tikzcd}[column sep=small]
 & b_1 \arrow[r, "2"] & b_2 \arrow[r, "1"] & b_3 \arrow[r, "1"] & b_4 \arrow[r, "2"] & b_5 \\
b_1 \arrow[d, "2"'] & b_1\otimes b_1 \arrow[r, dashed, "2"] & b_2\otimes b_1 \arrow[r, "1"] \arrow[d, dashed, "2"] & b_3\otimes b_1 \arrow[r, "1"] & b_4\otimes b_1 \arrow[r, dashed, "2"] & b_5\otimes b_1 \arrow[d, dashed, "2"] \\
b_2 \arrow[d, "1"'] & b_1\otimes b_2 \arrow[d, "1"] & b_2\otimes b_2 \arrow[r, "1"] & b_3\otimes b_2 \arrow[r, "1"] & b_4\otimes b_2 \arrow[d, "1"] & b_5\otimes b_2 \arrow[d, "1"] \\
b_3 \arrow[d, "1"'] & b_1\otimes b_3 \arrow[r, dashed, "2"] \arrow[d, "1"] & b_2\otimes b_3 \arrow[r, "1"] & b_3\otimes b_3 \arrow[d, "1"] & b_4\otimes b_3 \arrow[r, dashed, "2"] \arrow[d, "1"] & b_5\otimes b_3 \arrow[d, "1"] \\
b_4 \arrow[d, "2"'] & b_1\otimes b_4 \arrow[r, dashed, "2"] & b_2\otimes b_4 \arrow[d, dashed, "2"] & b_3\otimes b_4 \arrow[d, dashed, "2"] & b_4\otimes b_4 \arrow[r, dashed, "2"] & b_5\otimes b_4 \arrow[d, dashed, "2"] \\
b_5 & b_1\otimes b_5 & b_2\otimes b_5 \arrow[r, "1"] & b_3\otimes b_5 \arrow[r, "1"] & b_4\otimes b_5 & b_5\otimes b_5
\end{tikzcd}
$$

Three connected components with highest weight vectors:

$$
\begin{array}{llll}
 & b_1\otimes b_1, & b_1\otimes b_2, \text{ and} & b_1\otimes b_5 \\
\text{of wts:} & 2\bar{\Lambda}_2 & 2\bar{\Lambda}_2-\alpha_2=2\bar{\Lambda}_1 & 2\bar{\Lambda}_2-2\alpha_1-2\alpha_2=0.
\end{array}
$$

These three connected components are linked by 0-arrows. Since

$\tilde{e}_0(b_1\otimes b_5) = b_4\otimes b_5$ and $\tilde{e}_0(b_1\otimes b_1) = b_1\otimes b_4$ we conclude that

$$
H(b\otimes b') = \begin{cases} 2 & \text{if } b\otimes b' \in B(2\bar{\Lambda}_2) \\ 1 & \text{if } b\otimes b' \in B(2\bar{\Lambda}_1) \\ 0 & \text{if } b\otimes b' \in B(0) \end{cases}
$$

Using the energy function $H : B_\ell \otimes B_\ell \to \mathbb{Z}$ the affine weight of a path

$$P = \cdots \otimes p(k) \otimes \cdots \otimes p(2) \otimes p(1) \in P(\lambda, B_\ell)$$

is given by the following formula:

$$\begin{aligned} wt(p) = \lambda + \sum_{k=1}^{\infty} \big(wt(p(k)) - wt(b_k)\big) \\ - a_0^{-1}\delta\Big(\sum_{k=1}^{\infty} k\big(H(p(k+1)\otimes p(k)) - H(b_{k+1}\otimes b_k)\big)\Big). \end{aligned}$$

Ex: $\mathfrak{g} = C_2^{(1)}$

$$B_1 = B^{2,1} : \begin{tikzcd} b_1 \arrow[r, "2"] & b_2 \arrow[r, "1"] & b_3 \arrow[r, "1"] & b_4 \arrow[r, "2"] \arrow[lll, bend left, "0"] & b_5 \arrow[lll, bend right, "0"'] \end{tikzcd}$$

$$(P_{cl}^{+})_1 = \{\Lambda_0, \Lambda_1, \Lambda_2\}$$

Λ_0-ground state path: $\cdots \otimes b_5 \otimes b_1 \otimes b_5$

Λ_1-ground state path: $\cdots \otimes b_3 \otimes b_3 \otimes b_3$

Λ_2-ground state path: $\cdots \otimes b_1 \otimes b_5 \otimes b_1$

Now using signature rule we can draw

the path realization of the affine crystal $B(\Lambda_1)$ as follows:

$$
\begin{tikzcd}
 & & (\cdots b_3 b_3 b_3) \arrow[d, "1"] & & \\
 & & (\cdots b_3 b_3 b_4) \arrow[dl, "0"'] \arrow[dr, "2"] & & \\
 & (\cdots b_3 b_3 b_1) \arrow[d, "1"'] \arrow[dr, "2"] & & (\cdots b_3 b_3 b_5) \arrow[dl, "0"'] \arrow[d, "1"] & \\
 & (\cdots b_3 b_4 b_1) \arrow[d] & (\cdots b_3 b_3 b_2) \arrow[d, "1"] & (\cdots b_3 b_4 b_5) \arrow[d, "0"] & \\
 & (\cdots b_3 b_5 b_1) & (\cdots b_3 b_4 b_2) & (\cdots b_3 b_1 b_5) & \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

(Here we have dropped the tensor product symbol for convinience of presentation.)