$F = \mathbb{C}(q)$

$\mathcal{U}_q'(\mathfrak{g}) \subset \mathcal{U}_q(\mathfrak{g})$

$V$ irred. $\mathcal{U}_q'(\mathfrak{g})$-module

$V^{aff} = F[z, z^{-1}] \otimes_F V$

$\mathcal{U}_q(\mathfrak{g})$-module.

Fix $0 \neq \zeta \in F$. Define

$$J_\zeta = \langle z - \zeta \rangle \triangleleft F[z, z^{-1}]$$

Then
$$F[z, z^{-1}] / J_\zeta \cong F$$
$$z + J_\zeta \longmapsto \zeta$$

Define
$$V_\zeta = V^{aff}\Big|_{z=\zeta} \cong V^{aff} / J_\zeta V^{aff}$$
$$\cong F \otimes_{F[z,z^{-1}]} V^{aff}$$

as a $\mathcal{U}_q'(\mathfrak{g})$-module.

The $U_q'(\mathfrak{g})$-module action on $V_\zeta$ is defined by the homomorphism:

$$ev : U_q'(\mathfrak{g}) \longrightarrow \mathrm{End}_F V_\zeta$$

given by

$$ev(e_i) = \begin{cases} e_i & \text{if } i \neq 0 \\ \zeta e_0 & \text{if } i = 0 \end{cases}$$

$$ev(f_i) = \begin{cases} f_i & \text{if } i \neq 0 \\ \zeta^{-1} f_0 & \text{if } i = 0 \end{cases}$$

$$ev(K_i) = K_i \quad \forall\, i$$

Under this $U_q'(\mathfrak{g})$-module action

$$V_\zeta \cong V.$$

Suppose that the $U_q'(\mathfrak{g})$-module $V$ has crystal base $(L, B)$.

Then $V^{aff}$ has crystal base $(L^{aff}, B^{aff})$ where

$$L^{aff} = A_0[z, z^{-1}] \otimes_{A_0} L$$

$$B^{aff} = \{ b(m) := z^m \otimes b \mid m \in \mathbb{Z}, b \in B \} \subset L^{aff} / q L^{aff}$$

The action of Kashiwara operators $\tilde{e}_i, \tilde{f}_i$, $i \in I$ are given by

$$\tilde{e}_i(b(m)) = \begin{cases} (\tilde{e}_i b)(m) & \text{if } i \neq 0 \\ (\tilde{e}_0 b)(m+1) & \text{if } i = 0 \end{cases}$$

$$\tilde{f}_i(b(m)) = \begin{cases} (\tilde{f}_i b)(m) & \text{if } i \neq 0 \\ (\tilde{f}_i b)(m-1) & \text{if } i = 0 \end{cases}$$

Recall for $\zeta \in F$, $V_\zeta \cong V$. $V_\zeta$ is a $U_q'(\mathfrak{g})$-module called the evaluation module. The crystal base of $V_\zeta$ when $\zeta = 1$ coincides with $(L, B)$, crystal of $V$.

<u>Ex</u>: $\mathfrak{g} = \widehat{\mathfrak{sl}}(2)$

Consider the $\mathcal{U}_q'(\mathfrak{g})$-module

$$V = F v_0 \oplus F v_1 \quad , \quad F = \mathbb{C}(q)$$

with the action:

$$e_1 v_0 = 0 \ , \ e_1 v_1 = v_0$$

$$f_1 v_0 = v_1 \ , \ f_1 v_1 = 0$$

$$K_1 v_0 = q v_0 \ , \ K_1 v_1 = q^{-1} v_1$$

$$e_0 v_0 = v_1 \ , \ e_0 (v_1) = 0$$

$$f_0 v_0 = 0 \ , \ f_0(v_1) = v_0$$

$$K_0 v_0 = q^{-1} v_0 \ , \ K_0 v_1 = q v_1$$

$V$ has crystal base $(L, B)$ where

$$L = A_0 v_0 \oplus A_0 v_1 \ , \quad B = \{v_0, v_1\}$$

Crystal graph:

$$\begin{tikzcd} v_0 \arrow[r, bend left, "1"] & v_1 \arrow[l, bend left, "0"] \end{tikzcd}$$

$\cancel{\tilde{f}_0(v_0) = 0, \tilde{e}_0}$

$\tilde{f}_0(v_1) = v_0$ , $\tilde{f}_0(v_0) = 0$

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$V^{aff} = F[z, z^{-1}] \otimes_F V$ has crystal base $(L^{aff}, B^{aff})$ where

$$B^{aff} = \{ b(m) := z^m \otimes b \mid m \in \mathbb{Z}, b \in B \}$$
$$= \{ v_0(m), v_1(m) \mid m \in \mathbb{Z} \}$$

Crystal graph for $V^{aff}$ is

$$\cdots \xrightarrow{0} v_0(m+1) \xrightarrow{1} v_1(m+1) \xrightarrow{0} v_0(m) \xrightarrow{1} v_1(m) \xrightarrow{0} v_0(m-1) \xrightarrow{1} \cdots$$

The $U_q'(\mathfrak{g})$-module action on the evaluation module $V_\zeta$ is given by:

$e_1(v_0) = 0$, $e_1(v_1) = v_0$, $f_1(v_0) = v_1$, $f_1(v_1) = 0$

$K_1(v_0) = q v_0$, $K_1(v_1) = q^{-1} v_1$.

$e_0 v_0 = \varsigma v_1 \;,\; e_0 v_1 = 0$

$f_0 v_0 = 0 \;,\; f_0 v_1 = \varsigma^{-1} v_0$

$K_0 v_0 = q^{-1} v_0 \;,\; K_0 v_1 = q v_1$

$$\overline{wt}(v_0) = \varphi(v_0) - \varepsilon(v_0)$$
$$= \left(\varphi_0(v_0)\Lambda_0 + \varphi_1(v_0)\Lambda_1\right) - \left(\varepsilon_0(v_0)\Lambda_0 + \varepsilon_1(v_0)\Lambda_1\right)$$
$$= \left(0\Lambda_0 + 1\Lambda_1\right) - \left(1\Lambda_0 + 0\Lambda_1\right)$$
$$= \Lambda_1 - \Lambda_0$$

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$$= \left(1\Lambda_0 + 0\Lambda_1\right) - \left(0\Lambda_0 + 1\Lambda_1\right)$$
$$= \Lambda_0 - \Lambda_1$$

Note: level of $\overline{wt}(v_0)$ & $\overline{wt}(v_1)$

[redacted] $\overline{wt}(v_0)(c) = \overline{wt}(v_0)(h_0 + h_1)$

$= (\Lambda_1 - \Lambda_0)(h_0 + h_1) = 0.$

$\overline{wt}(v_1)(c) = (\Lambda_0 - \Lambda_1)(h_0 + h_1) = 0.$