$\mathfrak{g}$ affine Lie algebra / $\mathbb{C}$

$I = \{0, 1, \cdots, n\}$

$(A, \Pi, \check{\Pi}, P, \check{P})$ Cartan datum

$$A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad a_i \in \mathbb{Z}_{>0}$$

Assume $\gcd(a_0, a_1, \cdots, a_n) = 1$.

$\mathfrak{g}$ is one of following:

- Type I
  - $A_1^{(1)}$
  - $A_n^{(1)}$, $n \geq 2$
  - $B_n^{(1)}$, $n \geq 3$
  - $D_n^{(1)}$, $n \geq 4$
  - $C_n^{(1)}$, $n \geq 2$
  - $F_4^{(1)}$
  - $G_2^{(1)}$
  - $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$

Type II
- $A_2^{(2)}$
- $A_{2n}^{(2)}$, $n \geq 2$
- $A_{2n-1}^{(2)}$
- $D_{n+1}^{(2)}$
- $E_6^{(2)}$

Type III
- $D_4^{(3)}$

If $\mathfrak{g} \neq A_{2n}^{(2)}$, then $a_0 = 1$

If $\mathfrak{g} = A_{2n}^{(2)}$, then $a_0 = 2$.

$A = (a_{ij})_{(n+1)\times(n+1)}$ affine GCM

$\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$, $\check{\Pi} = \{h_0, h_1, \cdots, h_n\}$

$\mathfrak{h} = \text{span}\{h_0, h_1, \cdots, h_n, d\}$

$\Pi \subset \mathfrak{h}^*$

$\alpha_j(h_i) = a_{ij}$, $\alpha_j(d) = \delta_{j,0}$

$i \in I$

$\Lambda_i \in \mathfrak{h}^*$, $\Lambda_i(h_j) = \delta_{ij}$, $\Lambda_i(d) = 0$

$\delta = \sum a_i \alpha_i \in \mathfrak{h}^*$, called null root.

Weight lattice

$$P = \bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{1}{a_0}\delta$$

Co-weight lattice

$$\check{P} = \bigoplus_{i=0}^{n} \mathbb{Z}h_i \oplus \mathbb{Z}d$$

So $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$.

$A = (a_{ij})_{i,j \in I}$ affine GCM

A^T is also an affine GCM

$\overset{\vee}{\mathfrak{g}} = \mathfrak{g}(A^T)$ called Langland dual of $\mathfrak{g}$.

$A^T \begin{pmatrix} \check{a}_0 \\ \check{a}_1 \\ \check{a}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\gcd\{\check{a}_0, \cdots, \check{a}_n\} = 1$

$c = \sum_{i=0}^{n} \check{a}_i h_i \in Z(\mathfrak{g})$

$\text{span}\{c\} = Z(\mathfrak{g})$

c is called the canonical central element.

Δ = set of roots of $\mathfrak{g}$

$\Delta = \Delta_{re} \cup \Delta_{im}$

Δ_{re} = set of real roots
$= W\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$

where $W = \langle r_0, r_1, \cdots, r_n \rangle$ affine Weyl group.

Δ_{im} = set of imaginary roots
$= \mathbb{Z}\delta$ if $\mathfrak{g}$ is of type I.

A is symmetrizable

$D = \mathrm{diag}\{s_0, s_1, \cdots, s_n\}$, $s_i > 0$

so that DA is symmetric.

$\Rightarrow (\ , \ )$ nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$

$U_q(\mathfrak{g}) = \langle e_i, f_i, K_i^{\pm 1}, q^h \rangle$

associated with Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$

where $K_i = q^{s_i h_i} = q_i^{h_i}$, $q_i = q^{s_i}$

$U_q(\mathfrak{g})$ is called quantum affine algebra.

Define

$\bar{\check{P}} = \bigoplus_{i=0}^{n} \mathbb{Z} h_i \subsetneqq \check{P}$, classical dual weight lattice.

$\bar{P} = \bigoplus_{i=0}^{n} \mathbb{Z} \Lambda_i \subsetneqq P$, classical weight lattice.

Denote

$$\mathcal{U}_q'(\mathfrak{g}) = \text{subalgebra of } \mathcal{U}_q(\mathfrak{g}) \text{ gen. by } \{e_i, f_i, K_i^{\pm 1}\}$$

Then $\mathcal{U}_q'(\mathfrak{g})$ is the quantum group associated with the classical Cartan datum $(A, \Pi, \check{\Pi}, \bar{P}, \check{\bar{P}})$.

The main difference in the representation theory of $\mathcal{U}_q(\mathfrak{g})$ & $\mathcal{U}_q'(\mathfrak{g})$ is:

- All nontrivial irred $\mathcal{U}_q(\mathfrak{g})$-modules are infinite dimensional with finite dim'l weight spaces.
- The irred. $\mathcal{U}_q'(\mathfrak{g})$-modules can be finite dimensional. Furthermore, irred. infinite dim'l $\mathcal{U}_q'(\mathfrak{g})$-modules could have infinite dim'l weight spaces.

For $\lambda \in P$ (or $\bar{P}$), $\lambda(c)$ is called the level of λ.

Suppose V be a finite dim'l (irred.) $U_q'(\mathfrak{g})$-module with weight space decomposition:

$$V = \bigoplus_{\lambda \in \bar{P}} V_\lambda, \quad V_\lambda = \{v \in V \mid K_i v = q_i^{s_i \lambda(h_i)} v \ \forall i\}$$

$$q_i^{\langle \lambda, h_i \rangle} = q^{\langle \lambda, s_i h_i \rangle}$$

For an indeterminant z, set

$$V^{aff} = F[z, z^{-1}] \otimes_F V, \quad F = \mathbb{C}(q)$$

Define $U_q(\mathfrak{g})$ action on V^{aff} as:

For $z^m \otimes v \in V^{aff}$, define

$$e_0(z^m \otimes v) = z^{m+1} \otimes e_0 v$$

$$f_0(z^m \otimes v) = z^{m-1} \otimes f_0 v$$

$$e_i(z^m \otimes v) = z^m \otimes e_i v \quad , \quad i \neq 0$$
$$f_i(z^m \otimes v) = z^m \otimes f_i v \quad , \quad i \neq 0$$
$$K_i^{\pm 1}(z^m \otimes v) = z^m \otimes K_i^{\pm 1} v \, , \quad \forall i$$
$$q^d(z^m \otimes v) = q^{m a_0}(z^m \otimes v)$$

This action makes V^{aff} a $\mathcal{U}_q(\mathfrak{g})$-module called the affinization of V.

For $v \in V_\lambda$, $m \in \mathbb{Z}$, the affine weight of $z^m \otimes v$ is

$$\mathrm{wt}(z^m \otimes v) = \mathrm{aff}(\lambda) + m\delta \in P$$

where as the classical weight of $z^m \otimes v$ is

$$\overline{\mathrm{wt}}(z^m \otimes v) = \lambda \qquad \forall \; m \in \mathbb{Z}$$

Observe that $\overline{P} \cong P/\mathbb{Z}\delta$