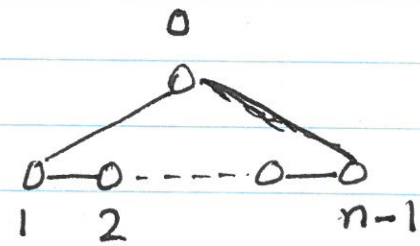
Affine Lie algebra:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & \vdots \\ \vdots & \vdots & & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix} = (a_{ij})_{i,j=0}^{n-1}$$

$n \times n$

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Dynkin diagram: 

$, = A_{n-1}^{(1)}$

$n > 2$

$I = \{0, 1, \cdots, n-1\}$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$ 

$= A_1^{(1)}$ $n = 2$

Cartan datum:

$\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$ lin. indep.

$\check{\Pi} = \{h_0, h_1, \cdots, h_{n-1}\}$

$\alpha_j(h_i) = a_{ij}$, $\alpha_j(d) = 0$ or 1

$\mathfrak{h} = \text{span}\{h_0, h_1, \cdots, h_{n-1}, d\}$

$\Pi \subseteq \mathfrak{h}^*$, $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$ null root.

$$\{\Lambda_i \mid 0 \leq i \leq n-1\} \subseteq \mathfrak{h}^*$$

by $\Lambda_j(h_i) = \delta_{ij}$, $\Lambda_j(d) = 0$.

$P = \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_{n-1} \oplus \mathbb{Z}\delta$

affine weight lattice.

$\check{P} = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_{n-1} \oplus \mathbb{Z}d$

affine dual weight lattice

The Lie algebra

$\mathfrak{g} = \mathfrak{g}(A)$ with Cartan datum

$\{A, \Pi, \check{\Pi}, P, \check{P}\}$

is called the affine Lie algebra $A_{n-1}^{(1)}$. It is generated by

$\{e_i, f_i \mid 0 \leq i \leq n-1\} \cup \check{P}$

satisfying the six Serre relations.

$c = h_0 + h_1 + \cdots + h_{n-1}$

is in $Z(\mathfrak{g})$.

$$\widehat{sl}(n)$$
$$\shortparallel$$
$$A^{(1)}_{n-1} \cong sl(n,\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d = g$$

$$[x\otimes t^i, y\otimes t^j] = [x,y]\otimes t^{i+j} + \delta_{i+j,0}\, tr(xy) ic$$

$$[c,g]=0 \quad , \quad [d,c]=0$$

$$[d, x\otimes t^i] = i\, x\otimes t^i \quad \left(\Rightarrow d = 1\otimes t\frac{d}{dt}\right)$$

$$e_i \rightarrow E_{i,i+1}\otimes 1, \quad f_i \longrightarrow E_{i+1,i}\otimes 1$$

$$h_i \longrightarrow (E_{ii} - E_{i+1,i+1})\otimes 1 \quad , \quad 1\le i\le n-1$$

$$e_0 \longrightarrow E_{n1}\otimes t^{1}, \quad f_0 \longrightarrow E_{1n}\otimes t^{-1}$$

$$h_0 \longrightarrow (E_{nn} - E_{11})\otimes 1 + c$$

$$[e_0, f_0] = [E_{n1}, E_{1n}]\otimes 1 + tr(E_{n1}E_{1n})c$$
$$= (E_{nn} - E_{11})\otimes 1 + c = h_0 .$$

Also $h_0 + h_1 + \cdots + h_{n-1}$

$$= \big((E_{nn} - E_{11}) + (E_{11} - E_{22}) + \cdots + (E_{n-1,n-1} - E_{nn})\big)\otimes 1$$
$$+ c$$
$$= c .$$

Consider the quantum group $U_q(\widehat{sl}(n))$ over $\mathbb{C}(q)$ with Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$.

$U_q(\widehat{sl}(n))$ is called a quantum affine algebra.

Consider the irreducible integrable module $V(\Lambda_0) \in \mathcal{O}^q_{int}$

Let $(L(\Lambda_0), B(\Lambda_0))$ be the crystal base for $V(\Lambda_0)$.

Want to realize $B(\Lambda_0)$ as "extended Young diagram."

Extended Young diagram:

$$Y = (y_k)_{k \geq 0}, \quad y_k \leq y_{k+1}, \quad y_k \in \mathbb{Z}$$

such that $\exists\, y_\infty \in \mathbb{Z}$, $0 \leq y_\infty \leq n-1$ and $y_k = y_\infty$ for $k >> 0$.

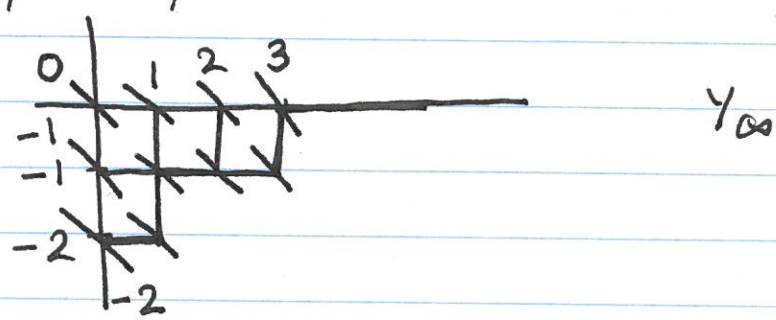
y_∞ is called the "charge" of Y.

Example:

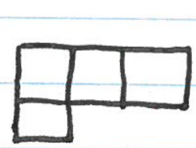
$$Y = (-2,-1,-1,0,0,\ldots) = (y_k)_{k\geq 0}$$

is an extended Young diagram of charge 0.

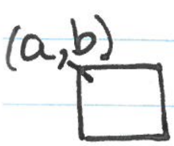
We can represent Y as a diagram in the 4th quadrant with y_k denoting the depth of the kth column.



The diagonal number of a corner at site (a,b) is $d = a+b$

$$Y \longmapsto ($$  $$, 0)$$

We can also represent Y by a colored Young diagram by coloring the box at ~~diagonal~~ site (a,b) by $(a+b) \bmod n = d \pmod n$.



So $Y = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ $\qquad -1 = n-1$

as a colored Young diagram with n colors: $0, 1, 2, \cdots, n-1$

Given an extended Young diagram $Y = (y_k)_{k \geq 0}$, we define

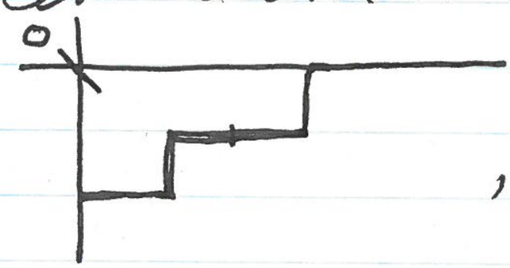
$$Y[n] = (y_k + n)_{k \geq 0}$$

which is again an extended Young diagram obtained from Y by giving a vertical shift of n units.

<u>Ex</u>: $Y = (-2, -1, -1, 0, \cdots)$ of charge 0

Then $Y[2] = (0, 1, 1, 2, 2, \cdots)$ of charge 2.

(Note that the charge has to be between 0 and $n-1$.)

$Y =$ 

, $Y[2] =$