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$\mathfrak{g} = sl(n, \mathbb{C})$,

$U_q(\mathfrak{g})$ = associated quantum group.

$\lambda \in P^+$

$V(\lambda)$ be the irred. $U_q(\mathfrak{g})$-module with highest weight λ.

$$\lambda = \sum_{i=1}^{n-1} k_i \Lambda_i, \quad k_i \in \mathbb{Z}_{\geq 0}, \quad \Lambda_i(h_j) = \delta_{ij}$$

$$V(\Lambda_1) = \text{span}_{\mathbb{C}(q)}\{v_1, v_2, \cdots, v_n\} = \mathbb{C}(q)^n.$$

called vector representation of $U_q(\mathfrak{g})$.

The crystal

$B(\Lambda_1) = \{v_1, v_2, \cdots, v_n\}$ with crystal graph

$$v_1 \xrightarrow{\tilde{f}_1} v_2 \xrightarrow{\tilde{f}_2} v_3 \longrightarrow \cdots \xrightarrow{\tilde{f}_{n-1}} v_n$$

Recall $\Lambda_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$, $\quad 1 \leq i \leq n-1$

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$wt(v_1) = \Lambda_1 = \varepsilon_1$

$wt(v_2) = \Lambda_1 - \alpha_1 = \varepsilon_1 - (\varepsilon_1 - \varepsilon_2) = \varepsilon_2$

$wt(v_3) = \Lambda_1 - \alpha_1 - \alpha_2 = \varepsilon_2 - \alpha_2 = \varepsilon_2 - (\varepsilon_2 - \varepsilon_3) = \varepsilon_3$

$\vdots$

$wt(v_n) = \varepsilon_n$

Consider $V(\Lambda_2)$

Then $V(\Lambda_2) \subseteq V(\Lambda_1) \otimes V(\Lambda_1)$

$B(\Lambda_1) = \{v_1, v_2, \cdots, v_n\}$

Recall $wt(v \otimes v') = wt(v) + wt(v')$

$v_1 \otimes v_2 \in V(\Lambda_1) \otimes V(\Lambda_1)$ is the highest weight vector of $V(\Lambda_2)$ with weight Λ_2.

Applying $\tilde{f}_i$, $1 \le i \le n-1$ to $v_1 \otimes v_2$ using the tensor product rule we can obtain the crystal $B(\Lambda_2)$ as a subset of $B(\Lambda_1) \otimes B(\Lambda_1)$.

In deed, for any $i \in I = \{1, 2, \cdots, n-1\}$,

$V(\Lambda_i)$ crystal $B(\Lambda_i)$ can be obtained as a subset of $B(\Lambda_1)^{\otimes i}$; i.e.

$$B(\Lambda_i) \subseteq \underbrace{B(\Lambda_1) \otimes \cdots \otimes B(\Lambda_1)}_{i \text{ times}}$$

For $\lambda = \sum_{i=1}^{n-1} k_i \Lambda_i \in P^+$, the crystal $B(\lambda)$ for $V(\lambda)$ can be obtained as a subset of

$$B(\Lambda_1)^{\otimes k_1} \otimes \big(B(\Lambda_1)\otimes B(\Lambda_1)\big)^{\otimes k_2} \otimes \cdots \otimes \big(\underbrace{B(\Lambda_1)\otimes\cdots\otimes B(\Lambda_1)}_{n-1 \text{ times}}\big)^{\otimes k_{n-1}}$$

$$= B(\Lambda_1)^{\otimes (k_1+2k_2+\cdots+(n-1)k_{n-1})}$$

$$= B(\Lambda_1)^{\otimes m}, \quad m = k_1+2k_2+\cdots+(n-1)k_{n-1}$$

$b \in B(\Lambda_1)^{\otimes m}$

$\Rightarrow b = b_1 \otimes b_2 \otimes \cdots \otimes b_m$

How we obtain $\tilde{f}_i b$?

We use signature rule:

Fix $i \in I$, $b = b_1 \otimes \cdots \otimes b_m \in B(\Lambda_1)^{\otimes m}$

For each $1 \le j \le m$, we assign the i-signature of b_j to be

$$\underbrace{- - \cdots -}_{\varepsilon_i(b_j)}\ \underbrace{+ + \cdots +}_{\varphi_i(b_j)}$$

i-signature of $b = b_1 \otimes b_2 \otimes \cdots \otimes b_m$ is

$$\underbrace{-\,-\cdots-}_{\varepsilon_i(b_1)}\underbrace{+\,+\cdots+}_{\varphi_i(b_1)}\cdots\cdots\underbrace{-\,-\cdots-}_{\varepsilon_i(b_m)}\underbrace{+\,+\cdots+}_{\varphi_i(b_m)}$$

We define the reduced i-signature of $b = b_1 \otimes \cdots \otimes b_m$ by removing $(+-)$ pairs sequencially. Then the reduced i-signature is of the form

$$\underbrace{-\,-\,-\cdots-}_{s}\underbrace{+\,+\cdots+}_{r}$$

Then $\tilde{f}_i$ acts on b_j corresponding to the left most $+$ in the reduced i-sig. Hence $\tilde{f}_i$ can be applied r times to $b = b_1 \otimes \cdots \otimes b_m$.

The $\tilde{e}_i$ acts on b_j corresponding to the right most $-$ in the reduced i-sig. Hence $\tilde{e}_i$ can be applied s times to $b = b_1 \otimes \cdots \otimes b_m$.

<u>Example:</u> $B(\Lambda_1)$: $v_1 \xrightarrow{1} v_2 \xrightarrow{2} \cdots \xrightarrow{n-1} v_n$

$$b = v_1 \otimes v_2 \otimes v_2 \otimes v_3 \otimes v_5 \in B(\Lambda_1)^{\otimes 5}$$

$i = 2$:

2-signature of b is

$$\cdot\ \cdot\cdot + \cdot + - \cdot\ \cdot\ \cdot$$

reduced 2-signature is

$+$

Hence $\tilde{e}_2(b) = 0$

$$\tilde{f}_2(b) = v_1 \otimes \tilde{f}_2 v_2 \otimes v_2 \otimes v_3 \otimes v_5$$
$$= v_1 \otimes v_3 \otimes v_2 \otimes v_3 \otimes v_5 .$$