Ex(1).

$$V(m) = \operatorname{span}_{\mathbb{C}(q)} \{v_0, v_1, \cdots, v_m\}$$

$$V(n) = \operatorname{span}_{\mathbb{C}(q)} \{u_0, u_1, \cdots, u_n\}$$

be irreducible $U_q(sl(2))$ modules.

Assume $m \geq n$.

Crystal graphs:

$$V(m): \quad v_0 \xrightarrow{\tilde{f}} v_1 \xrightarrow{\tilde{f}} v_2 \longrightarrow \cdots \xrightarrow{\tilde{f}} v_m \quad (B(m))$$

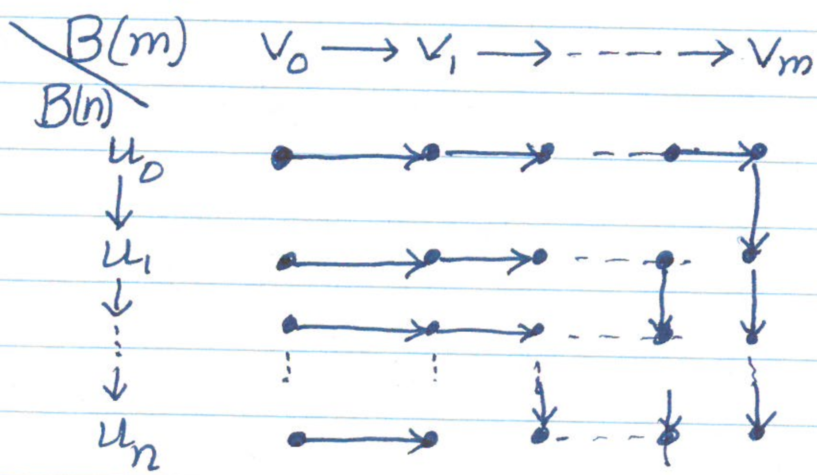
$$V(n): \quad u_0 \xrightarrow{\tilde{f}} u_1 \xrightarrow{\tilde{f}} u_2 \longrightarrow \cdots \xrightarrow{\tilde{f}} u_n \quad (B(n))$$

$V(m) \otimes V(n)$ is a reducible $U_q(sl(2))$ module.

$V(m) \otimes V(n)$ has crystal

$$B(m) \otimes B(n) = \{v_i \otimes u_j \mid 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

Crystal graph of $V(m) \otimes V(n)$:



$$\Rightarrow B(m)\otimes B(n) = B(m+n)\oplus B(m+n-2)\oplus \cdots$$
$$\oplus\, B(m-n)$$

$$\Rightarrow V(m)\otimes V(n) \cong \bigoplus_{k=0}^{n} V(m+n-2k)$$

which is the q-Clebsh-Gordon formula.

<u>Ex(2)</u> $V(\Lambda_1) = \mathbb{C}(q)^3$ irred. $U_q(sl(3))$-module

$$V(\Lambda_1) = \operatorname{span}_{\mathbb{C}(q)}\{v_1, v_2, v_3\}$$

Crystal for $V(\Lambda_1)$:

$$B(\Lambda_1):\quad \begin{array}{ccccc} v_1 & \xrightarrow{\tilde{f}_1} & v_2 & \xrightarrow{\tilde{f}_2} & v_3 . \\ \shortparallel & & \shortparallel & & \shortparallel \\ \boxed{1} & \xrightarrow{1} & \boxed{2} & \xrightarrow{2} & \boxed{3} \end{array}$$

The crystal for $V(\Lambda_1)\otimes V(\Lambda_1)$ is:

$$\begin{tikzcd}
B(\Lambda_1) \backslash B(\Lambda_1) & \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \\
\boxed{1} \arrow[d, "1"'] & \boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{2}\otimes\boxed{1} \arrow[r, "2"] \arrow[d, "1"] & \boxed{3}\otimes\boxed{1} \arrow[d, "1"] \\
\boxed{2} \arrow[d, "2"'] & \boxed{1}\otimes\boxed{2} \arrow[d, "2"] & \boxed{2}\otimes\boxed{2} \arrow[r, "2"] & \boxed{3}\otimes\boxed{2} \arrow[d, "2"] \\
\boxed{3} & \boxed{1}\otimes\boxed{3} \arrow[r, "1"] & \boxed{2}\otimes\boxed{3} & \boxed{3}\otimes\boxed{3}
\end{tikzcd}$$

$$wt\left(\boxed{1}\otimes\boxed{1}\right) = 2\Lambda_1$$

$$wt\left(\boxed{1}\otimes\boxed{2}\right) = 2\Lambda_1 - \alpha_1 = \Lambda_2$$

$$(2\Lambda_1 - \alpha_1)(h_1) = 0 \ , \quad (2\Lambda_1 - \alpha_1)(h_2) = 1$$

$$\Longrightarrow \quad V(\Lambda_1) \otimes V(\Lambda_1) \cong V(2\Lambda_1) \oplus V(\Lambda_2)$$

Crystal for $V(2\Lambda_1)$ is:

$$\begin{tikzcd}
& \boxed{1}\otimes\boxed{1} \arrow[d, "1"] & \\
& \boxed{2}\otimes\boxed{1} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\boxed{2}\otimes\boxed{2} \arrow[dr, "2"] & & \boxed{3}\otimes\boxed{1} \arrow[dl, "1"'] \\
& \boxed{3}\otimes\boxed{2} \arrow[d, "2"] & \\
& \boxed{3}\otimes\boxed{3} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \boxed{1\,|\,1} \arrow[d, "1"] & \\
& \boxed{1\,|\,2} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\boxed{2\,|\,2} \arrow[dr, "2"] & & \boxed{1\,|\,3} \arrow[dl, "1"'] \\
& \boxed{2\,|\,3} \arrow[d, "2"] & \\
& \boxed{3\,|\,3} &
\end{tikzcd}$$

Crystal for $V(\Lambda_2)$ is:

$$\begin{tikzcd}
\boxed{1}\otimes\boxed{2} \arrow[d, "2"] \\
\boxed{1}\otimes\boxed{3} \arrow[d, "1"] \\
\boxed{2}\otimes\boxed{3}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \arrow[d, "2"] \\
\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \arrow[d, "1"] \\
\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}
\end{tikzcd}$$

Ex(3) $V(\Lambda_1)$ and $V(\Lambda_2)$ are irreducible $U_q(sl(2))$-module with crystal graphs

$$V(\Lambda_1): \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \qquad (B(\Lambda_1))$$

$$V(\Lambda_2): \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$

Crystal graph for $V(\Lambda_1) \otimes V(\Lambda_2)$:

$$\begin{tikzcd}
\diagdown^{B(\Lambda_1):}_{B(\Lambda_2)} & \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \\
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \arrow[d, "2"] & \boxed{1}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \arrow[r, "1"] \arrow[d, "2"'] & \boxed{2}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \arrow[r, "2"] & \boxed{3}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \arrow[d, "2"] \\
\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \arrow[d, "1"] & \boxed{1}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \arrow[r, "1"] & \boxed{2}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \arrow[d, "1"] & \boxed{3}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \arrow[d, "1"] \\
\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} & \boxed{1}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} & \boxed{2}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \arrow[r, "2"] & \boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}
\end{tikzcd}$$

$$wt\left(\boxed{1}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right) = \Lambda_1 + \Lambda_2$$

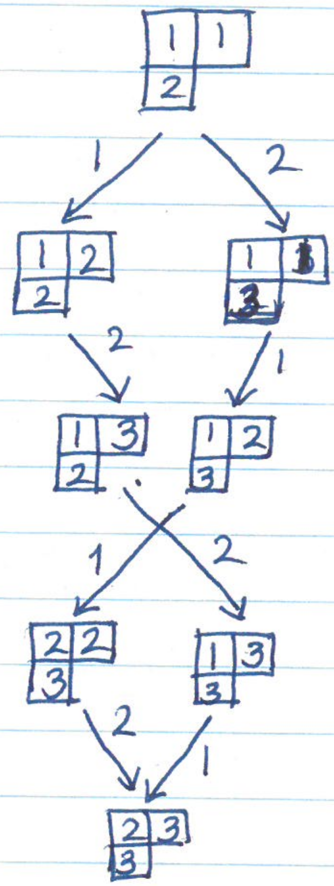
$$wt\left(\boxed{1}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right) = \Lambda_1 + \Lambda_2 - \alpha_1 - \alpha_2 = 0$$

$$(\Lambda_1 + \Lambda_2 - \alpha_1 - \alpha_2)(h_1) = 0$$

$$(\Lambda_1 + \Lambda_2 - \alpha_1 - \alpha_2)(h_2) = 0$$

$$\Rightarrow V(\Lambda_1) \otimes V(\Lambda_2) \cong V(\Lambda_1 + \Lambda_2) \oplus V(0)$$

Crystal graph for $V(\Lambda_1 + \Lambda_2)$:



Crystal graph for $V(0)$: