$V^q \in \mathcal{O}^q_{int}$

B crystal for V^q

$\tilde{e}_i, \tilde{f}_i : B \longrightarrow B \cup \{0\}$, $i \in I$

$\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}$

$wt : B \longrightarrow P$

Ex(1) $\lambda \in P^+$, irred. $V^q(\lambda)$. $V^q(\lambda)$ has crystal base $(L(\lambda), B(\lambda))$.

$$L(\lambda) = \sum_{\substack{k \geq 0 \\ i_j \in I}} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda$$

$$B(\lambda) = \{ b \in L(\lambda)/qL(\lambda) \mid b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda \neq 0 \bmod qL(\lambda) \}$$

$$\tilde{e}_i, \tilde{f}_i : B(\lambda) \longrightarrow B(\lambda) \cup \{0\}$$

$\varepsilon_i(b) = \max \{ k \mid \tilde{e}_i^k b \neq 0 \}$, $b \in B(\lambda)$

$= \#$ of i-arrows coming into b.

$\varphi_i(b) = \max \{ k \mid \tilde{f}_i^k b \neq 0 \}$

$= \#$ of i-arrows going out of b.

$$wt(b) = \sum_{i \in I} (\varphi_i(b) - \varepsilon_i(b))\Lambda_i \,, \quad b \in B(\lambda)$$

$$\varepsilon_i, \varphi_i : B(\lambda) \longrightarrow \mathbb{Z} \cup \{-\infty\}$$

$$wt : B(\lambda) \longrightarrow P$$

$(B(\lambda), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, wt, i \in I)$ is a combinatorial crystal.

<u>Ex(2)</u> Fix $\lambda \in P$. Consider $T_\lambda = \{t_\lambda\}$.

For $i \in I$, define

$$\tilde{e}_i(t_\lambda) = 0 \,, \quad \tilde{f}_i(t_\lambda) = 0$$

$$\varepsilon_i(t_\lambda) = -\infty \,, \quad \varphi_i(t_\lambda) = -\infty$$

$$wt(t_\lambda) = \lambda$$

$\{T_\lambda, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, wt\}$ is a (combinatorial) crystal. (exer.)

<u>Ex(3)</u> Fix $i \in I$. Consider $B^{(i)} = \{b_i(n) \mid n \in \mathbb{Z}\}$

For $j \in I$, $n \in \mathbb{Z}$ define

$$\tilde{e}_j(b_i(n)) = \begin{cases} b_i(n+1) & , \text{ if } j = i \\ 0 & , \text{ if } j \neq i \end{cases}$$

$$\tilde{f}_j(b_i(n)) = \begin{cases} b_i(n-1) & , \text{ if } j=i \\ 0 & , \text{ if } j \neq i \end{cases}$$

$$\varepsilon_j(b_i(n)) = \begin{cases} -n & , \text{ if } j=i \\ -\infty & , \text{ if } j \neq i \end{cases}$$

$$\varphi_j(b_i(n)) = \begin{cases} n & , \text{ if } j=i \\ -\infty & , \text{ if } j \neq i \end{cases}$$

$$wt(b_i(n)) = n\alpha_i .$$

$\{B^{(i)}, \tilde{e}_j, \tilde{f}_j, \varepsilon_j, \varphi_j, wt\}$ is a crystal.

For example,

$$\varphi_j(b_i(n)) \overset{?}{=} \varepsilon_j(b_i(n)) + \langle h_j, wt(b_i(n))\rangle$$

$j \neq i$, true

$j = i$: LHS $= n$,

RHS $= -n + \langle h_i, n\alpha_i \rangle$

$= -n + n\alpha_i(h_i) = -n + 2n = n =$ LHS.

$$wt(\tilde{e}_j(b_i(n))) \overset{?}{=} wt(b_i(n)) + \alpha_i \text{ if } \tilde{e}_j(b_i(n)) \in B^{(i)}$$

$\Rightarrow j=i$: LHS $= wt(b_i(n+1)) = (n+1)\alpha_i$

RHS $= n\alpha_i + \alpha_i$ $=$ LHS .

Ex(4) For $\lambda \in P$, let $M^q(\lambda)$ be the Verma module for $U_q(sl(2))$.
For $\lambda = 0$, $M^q(0)$ is the $U_q(sl(2))$ verma module with weight 0 and highest weight vector u_0. Then

$$M(0) = \text{span}\{f^{(k)}u_0 \mid k \geq 0\} \quad (\mathbb{C}(q)\text{ span})$$
$$\cong U_q^-(sl(2)) \quad (\text{as vector space})$$

where $u_0 \longrightarrow 1$.

Consider set $B(\infty) = \{f^{(k)}u_0 \mid k \geq 0\}$

Define

$$\tilde{e}(f^{(k)}u_0) = f^{(k-1)}u_0, \quad \tilde{f}(f^{(k)}u_0) = f^{(k+1)}u_0$$
$$\varepsilon(f^{(k)}u_0) = k, \quad \varphi(f^{(k)}u_0) = -k$$
$$wt(f^k u_0) = -2k$$

$(B(\infty), \tilde{e}, \tilde{f}, \varepsilon, \varphi, wt)$ is a crystal for $U_q(sl(2))$. (exer.)

Define $L(\infty) = \sum_{k \geq 0} A_0 f^{(k)} u_0$

Then $(L(\infty), B(\infty))$ is the crystal base for $M(0)$, hence $U_q^-(sl(2))$.

$\mathfrak{g}$ symmetrizable Kac-Moody alg. / $\mathbb{C}$ and $U_q(\mathfrak{g})$ be the associated quantum group. Let $V^q, W^q \in \mathcal{O}^q_{int}$ with crystal bases (L_1, B_1) and (L_2, B_2) respectively. Set

$L = L_1 \otimes_{A_0} L_2$ and

$B = \{(b_1, b_2) \mid b_1 \in B_1, b_2 \in B_2\} := B_1 \otimes B_2$

Then (L, B) is the crystal base for $V^q \otimes W^q$ with the maps $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, wt$, $i \in I$ given by: For $(b_1 \otimes b_2) \in B$,

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle\}$$

$$\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle\}$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2).$$