$V^q \in \mathcal{O}^q_{int}$, (L, B) be the crystal base for V^q.

Then for $b, b' \in B$,

$$\tilde{f}_i b = b' \iff \tilde{e}_i b' = b.$$

B is called the <u>crystal</u> for V^q.

<u>Crystal graph:</u>

Every $b \in B$ is a node in the graph and we draw an i-arrow

$$b \xrightarrow{i} b'$$

if $\tilde{f}_i b = b'$.

- The crystal graph of V^q is connected $\iff V^q$ is irreducible.

Let $\lambda \in wt(V^q)$. Then

$$\dim V^q_\lambda = \# B_\lambda = mult(\lambda)$$

$$\Longrightarrow \quad ch(V^q) = \sum_{\lambda \in wt(V^q)} (\# B_\lambda)\, e(\lambda).$$

<u>Thm</u>: $V^q, W^q \in \mathcal{O}^q_{int}$, $\psi : V^q \longrightarrow W^q$ isom. and (L, B) crystal base for V^q. Then $(\psi(L), \psi(B))$ is the crystal base for W^q.

<u>Defn</u>: $V^q \in \mathcal{O}^q_{int}$ and (L, B) crystal base for V^q. Suppose (L', B') is another crystal base for V^q.

Then we say (L, B) is isom. to (L', B') if there is a A_0-linear isom. $\psi : L \longrightarrow L'$ such that

(1) $\forall\, i \in I$, $\tilde{e}_i \psi = \psi \tilde{e}_i$, $\tilde{f}_i \psi = \psi \tilde{f}_i$

(2) $\psi : L/qL \longrightarrow L'/qL'$ defines a bijection $\psi : B \cup \{0\} \longrightarrow B' \cup \{0\}$ with $\psi \tilde{e}_i(b) = \tilde{e}_i \psi(b)$, $\psi \tilde{f}_i(b) = \tilde{f}_i \psi(b)$ $\forall\, i \in I$, $b \in B$.

<u>Thm</u>: $V^q \in \mathcal{O}^q_{int}$. Then the crystal base (L, B) is ~~isomorphic~~ unique up to isomorphism.

<u>Thm</u> $V^q, W^q \in \mathcal{O}^q_{int}$ and (L_1, B_1), (L_2, B_2) crystal bases for V^q and W^q respectively. Then

$$(L_1 \oplus L_2, \underbrace{B_1 \oplus B_2}_{= B_1 \cup B_2}) \text{ is a crystal base for } V^q \oplus W^q.$$

Conversely, suppose $V^q \in \mathcal{O}^q_{int}$ with crystal base (L, B) and suppose $V^q = V_1^q \oplus V_2^q$, $V_1^q, V_2^q \in \mathcal{O}^q_{int}$. Suppose there is A_0-submodules $L_1 \subset V_1^q$, $L_2 \subset V_2^q$ of L and subsets $B_1 \subset L_1/qL_1$, $B_2 \subset L_2/qL_2$ such that $L = L_1 \oplus L_2$, $B = B_1 \cup B_2$, then (L_1, B_1), (L_2, B_2) are crystal bases for V_1^q, V_2^q resp.

<u>Thm:</u> $V^q \in \mathcal{O}^q_{int}$, $V^q = \bigoplus_{\lambda\in P} V^q_\lambda$ and (L, B) crystal base for V^q. For fixed $i \in I$, $u \in V^q_\lambda$,

$$u = \sum_{k=0}^{N} f_i^{(k)} u_k, \quad u_k \in V^q_{\lambda+k\alpha_i} \cap \ker\{e_i\}.$$

Note $(\lambda + k\alpha_i)(h_i) = \lambda(h_i) + 2k$. Suppose $\lambda(h_i) + 2k \geq k \geq 0$. Then

(1) $u_k \in L$, $k \geq 0$

(2) If $\tilde{e}_i u \in qL$, then $u_k \in qL$, $1 \leq k \leq N$ $(\Rightarrow u = u_0 \pmod{qL})$.

(3) If $u + qL \in B = L/qL$, then there is $0 \leq k_0 \leq N$ such that

(i) $u_k \in qL \quad \forall\ k \neq k_0$

(ii) $u_{k_0} + qL \in B$

(iii) $u = f_i^{(k_0)} u_{k_0} \pmod{qL}$

$(\Rightarrow \tilde{e}_i u = f_i^{(k_0-1)} u_{k_0} \pmod{qL}.)$

$V^q \in \mathcal{O}^q_{int}$, $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ and

(L, B) crystal base.

For $i \in I$, $b \in B_\lambda$, $\lambda \in wt(V^q)$

Define

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in B\}$$

$=$ # of i-arrows coming to b

(in the crystal graph of B)

$$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\}$$

$=$ # of i-arrows going out of b.

$$\underbrace{\bullet \xrightarrow{i} \bullet \cdots \bullet \xrightarrow{i} \bullet \xrightarrow{i}}_{\varepsilon_i(b)} \underset{b}{\bullet} \underbrace{\xrightarrow{i} \bullet \xrightarrow{i} \bullet \cdots \bullet \xrightarrow{i} \circ}_{\varphi_i(b)}$$

By $U_q(sl(2))$ representation theory

$$\varphi_i(b) - \varepsilon_i(b) = wt(b)(h_i)$$

$$= \langle h_i, \underbrace{wt(b)}_{= \lambda} \rangle$$

If $\tilde{e}_i b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$$

If $\tilde{f}_i b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$$

Note $wt(\tilde{e}_i b) = wt(b) + \alpha_i$ if $\tilde{e}_i b \in B$

$wt(\tilde{f}_i b) = wt(b) - \alpha_i$ if $\tilde{f}_i b \in B$

Abstracting the properties of crystal base we define the combinatorial crystal B for $V^q \in \mathcal{O}^q_{int}$ as follows:

<u>Crystal</u> is a nonempty set B equipped with maps:

$$wt : B \longrightarrow P$$

$$\left.\begin{array}{l} \tilde{e}_i, \tilde{f}_i : B \longrightarrow B \cup \{0\} \\ \varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\} \end{array}\right\} \forall i \in I$$

satisfying

1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle$

2) $wt(\tilde{e}_i b) = wt(b) + \alpha_i$ if $\tilde{e}_i b \in B$

3) $wt(\tilde{f}_i b) = wt(b) - \alpha_i$ if $\tilde{f}_i b \in B$

4) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in B$

5) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in B$.

6) For $b, b' \in B$, $\tilde{f}_i b = b' \Leftrightarrow \tilde{e}_i b' = b$.

7) If $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = 0$ and $\tilde{f}_i b = 0$
