$U_q(\mathfrak{g})$ quantum group over $F = \mathbb{C}(q)$

Consider subring $A_0 \subseteq F$ to be

$$A_0 = \left\{ \frac{g(q)}{h(q)} \,\middle|\, g(q), h(q) \in \mathbb{C}[q],\ h(0) \neq 0 \right\}$$

Exer. A_0 is an integral domain

Furthermore,

A_0 has fraction field $\mathbb{C}(q)$

Defn: $V^q \in \mathcal{O}^q_{int}$

A free A_0-submodule L of V^q is a crystal lattice if

(1) $\mathbb{C}(q) \otimes_{A_0} L \cong V^q$

(2) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap V^q_\lambda$

(3) For each $i \in I$,

$\tilde{e}_i(L) \subseteq L$, $\tilde{f}_i(L) \subset L$

<u>Ex(1)</u> $U_q(sl(3))$, $V^q = \mathbb{C}(q)^3 = V^q(\Lambda_1)$

$$\parallel$$

$$\mathrm{span}_{\mathbb{C}(q)}\{v_1, v_2, v_3\}$$

$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. $v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_3$

v_1 = highest wt. vector with highest wt Λ_1

$$\Rightarrow q^{h_i} v_1 = q^{\Lambda_1(h_i)} v_1 = \begin{cases} q v_1, & i=1 \\ v_1, & i=2 \end{cases}$$

$$v_2 = f_1 v_1 = f_1^{(1)} v_1, \quad v_3 = f_2 v_2 = f_2^{(1)} v_2$$

$$\mathrm{wt}(V^q) = \{\Lambda_1, \Lambda_1 - \alpha_1, \Lambda_1 - \alpha_1 - \alpha_2\}$$

$$u \in V^q_{\Lambda_1} \Rightarrow u = \alpha v_1 = u_0 + 0 + 0 \cdots$$
$$= u_0$$

$$v \in V^q_{\Lambda_1 - \alpha_1}, \quad v = u_0 + f_1^{(1)} u_1 + \cdots$$

$$\mathrm{wt}(u_0) = \Lambda_1 - \alpha_1 \Rightarrow u_0 = \beta v_2$$

$$\mathrm{wt}(u_1) = (\Lambda_1 - \alpha_1) + \alpha_1 = \Lambda_1 \Rightarrow u_1 = \gamma v_1$$

$$\mathrm{wt}(u_2) = (\Lambda_1 - \alpha_1) + 2\alpha_1 \notin \mathrm{wt}(V^q) \Rightarrow u_2 = 0$$

and similarly $u_3 = u_4 = \cdots = 0$.

$$\tilde{e}_1(v) = \sum_{k \geq 1} f_1^{(k-1)} u_k = u_1 = \beta v_1$$

$$\tilde{f}_1(v) = \sum_{k \geq 0} f_1^{(k+1)} u_k = \underbrace{f_1^{(1)} u_0}_{=\,0} + \underbrace{f_1^{(2)} u_1}_{=\,\frac{1}{[2]} f_1^2 u_1 \,=\, 0} + \cdots$$

$$\begin{tikzcd} v_1 \arrow[r, "\tilde{f}_1"] & v_2 \arrow[l, bend left, "\tilde{e}_1"] \arrow[r, "\tilde{f}_2"] & v_3 \arrow[l, bend left, "\tilde{e}_2"] \end{tikzcd}$$

$L = A_0 v_1 \oplus A_0 v_2 \oplus A_0 v_3$ is the crystal lattice for $V^q(\Lambda_1)$.

Take $\quad J_0 = \langle q \rangle \triangleleft A_0$

J_0 is a maximal ideal of A_0.

Then $\quad A_0/J_0$ is a field and

$$A_0/J_0 \cong \mathbb{C}$$

$$f(q) + J_0 \longmapsto f(0)$$

Recall L is a A_0-submodule of V^q.

$J_0 L$ is a A_0-submodule of L

$$L/J_0L = L/qL \cong \mathbb{C} \otimes_{A_0} L$$

$$\tilde{e}_i(L) \subseteq L \Rightarrow \tilde{e}_i : L/qL \longrightarrow L/qL$$

i.e. $\tilde{e}_i(L/qL) \subseteq \hat{e}_i(L/qL)$

$$\tilde{f}_i(L) \subseteq L \Rightarrow \tilde{f}_i : L/qL \longrightarrow L/qL$$

i.e. $\tilde{f}_i(L/qL) \subseteq L/qL$.

$$\mathbb{C} \otimes_{A_0} L \cong L/qL$$

$$\parallel$$

$$\mathrm{span}_{\mathbb{C}}\{L\}$$

$$v \longmapsto \bar{v} \quad \text{(i.e. } q \to 0$$

$$\parallel$$

$$\bar{v} + qL$$

called the crystal limit.)

<u>Defn</u> (Crystal Base) The crystal base of $V^q \in \mathcal{O}^q_{int}$ is a pair (L, B) satisfying the following conditions:

(1) L is a crystal lattice for V^q.

(2) B is a $\mathbb{C}$-basis of L/qL

(3) $B = \bigcup_{\lambda \in P} B_\lambda$, $B_\lambda = B \cap L_\lambda / qL_\lambda$

(4) $\tilde{e}_i(B) \subseteq B \sqcup \{0\}$, $\tilde{f}_i(B) \subseteq B \sqcup \{0\}$ for each $i \in I$.

(5) For $b, b' \in B$ and any $i \in I$,

$$\tilde{f}_i b = b' \iff \tilde{e}_i b' = b.$$