$\mathfrak{g}$ = symmetrizable Kac-Moody algebra/$\mathbb{C}$.

$U_q(\mathfrak{g})$ = associated quantum group.

$I = \{1, 2, \cdots, n\}$ index set

$A = (a_{ij})_{i,j \in I}$ symmetrizable GCM

$(\, , \,) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ nondeg. invariant bilinear form.

Let $V^q \in \mathcal{O}^q_{int}$ $U_q(\mathfrak{g})$-module

For fixed $i \in I$,

$$U_q(\mathfrak{g})_{(i)} = \text{subalg. generated by } \{e_i, f_i, t_i = q_i^{h_i}, t_i^{-1}\}$$
$$\cong U_q(sl(2))$$

Recall that, as $U_q(\mathfrak{g})_{(i)}$-module

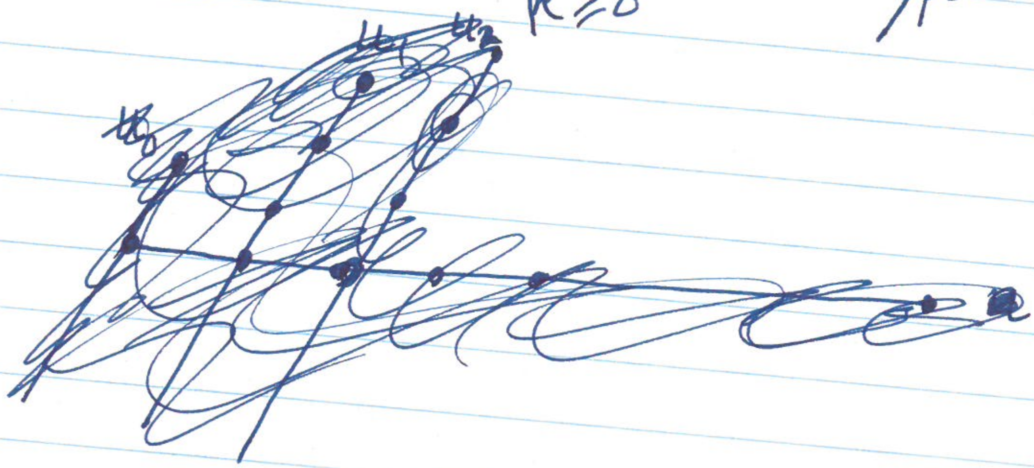
$$V^q = \bigoplus_{k \geq 0} c_k V^q(k) \quad , \quad c_k \in \mathbb{Z}_{\geq 0}$$

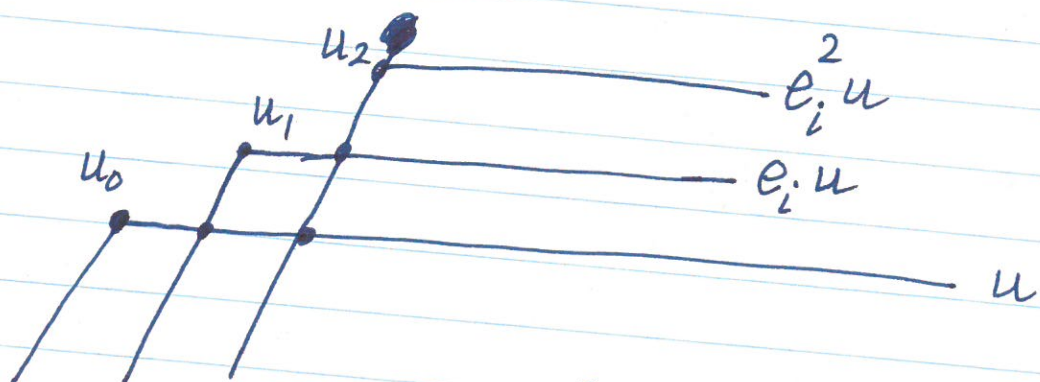
where $V^q(k)$ is irred. $U_q(\mathfrak{g})_{(i)}$-~~mod~~ module of dimension $k+1$.

$$V^q \in \mathcal{O}^q_{int} \Rightarrow V^q = \bigoplus_{\mu \in wt(V^q)} V^q_\mu ,$$

$$V^q_\mu = \{ v \in V^q \mid q^h \cdot v = q^{\mu(h)} v \ \forall h \in \check{P} \}. \qquad \dim V^q_\mu < \infty .$$

Let $u \in V^q_\mu = \left(\bigoplus_{k \geq 0} c_k V^q(k) \right)_\mu$





For each $i \in I$, $u \in V^q_\mu$ can be uniquely written in the form

$$u = u_0 + f_i^{(1)} \cdot u_1 + f_i^{(2)} \cdot u_2 + \cdots + f_i^{(N)} \cdot u_N$$

$$= \sum_{k=0}^{N} f_i^{(k)} u_k \quad \text{for some } N \in \mathbb{Z}_{\geqslant 0}$$

where $u_k \in V_{\mu + k\alpha_i} \cap \ker\{e_i\}$

Observe that $u_k \neq 0 \Leftrightarrow \mu(h_i) + k \geqslant 0$

We define the Kashiware operators

$$\tilde{e}_i, \tilde{f}_i : V^q \longrightarrow V^q \quad \text{by}$$

for $u \in V^q_\mu$, $u = \sum_{k=0}^{N} f_i^{(k)} u_k$

$$\tilde{e}_i(u) = \sum_{k=\cancel{0}\,1}^{N} f_i^{(k-1)} u_k$$

and $$\tilde{f}_i(u) = \sum_{k=0}^{N} f_i^{(k+1)} u_k$$

$\tilde{e}_i$ and $\tilde{f}_i$ is defined on V^q by linearity.

Prop: Let $V^q, W^q \in \mathcal{O}^q_{int}$ and

$\varphi: V^q \longrightarrow W^q$ be any $U_q(\mathfrak{g})$-module homomorphism. (i.e. φ preserves the weight spaces and commutes with $U_q(\mathfrak{g})$ action).
Then for each $i \in I$, we have

$$\tilde{e}_i \varphi = \varphi \tilde{e}_i \text{ and } \tilde{f}_i \varphi = \varphi \tilde{f}_i .$$

Proof: Let $u \in V^q_\mu$. Then $\varphi(u) \in W^q_\mu$.
For $i \in I$
Let $u = \sum_{k \geq 0} f_i^{(k)} u_k$ be the unique ~~expression~~ expression with $u_k \in V^q_{\mu + k\alpha_i} \cap \ker(e_i)$.

$$\Rightarrow \tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k$$

$$\Rightarrow \varphi(\tilde{e}_i u) = \sum_{k \geq 1} \varphi(f_i^{(k-1)} u_k)$$

$$= \sum_{k \geq 1} f_i^{(k-1)} \varphi(u_k) \text{ since } \varphi \text{ is hom.}$$

$$\varphi(u) = \sum_{k \geq 0} \varphi(f_i^{(k)} u_k) = \sum_{k \geq 0} f_i^{(k)} \varphi(u_k)$$

where $\varphi(u_k) \in W^q_{\mu + k\alpha_i} \cap \ker(e_i)$ since φ is hom.

Hence $\tilde{e}_i \varphi(u) = \sum_{k \geq 1} f_i^{(k-1)} \varphi(u_k) = \varphi(\tilde{e}_i u)$

$\Rightarrow \tilde{e}_i \varphi = \varphi \tilde{e}_i$. Similarly, $\tilde{f}_i \varphi = \varphi \tilde{f}_i$. //