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$\lambda \in P^+$

$V^q(\lambda)$ = irred. highest weight $U_q(\mathfrak{g})$-module with highest weight vector v_λ.

Then we have

- $V^q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$
- $q^h \cdot v_\lambda = q^{\lambda(h)} v_\lambda \quad \forall\, h \in P^\vee$
- $e_i \cdot v_\lambda = 0 \quad \forall\, i \in I$
- $f_i^{\lambda(h_i)+1} \cdot v_\lambda = 0 \quad \forall\, i \in I$.

~~Thm~~ $\lambda \in P$, $V^q(\lambda) = M^q(\lambda) / N^q(\lambda)$ is a irred. highest wt. $U_q(\mathfrak{g})$-module with highest weight λ.

Thm: For $\lambda \in P$,

$$V^q(\lambda) \in \mathcal{O}^q_{int} \iff \lambda \in P^+.$$

Lusztig's Thm: (1988)

For $\lambda \in P^+$, let $V^q(\lambda)$ (resp. $V(\lambda)$) be the irred. highest weight $U_q(\mathfrak{g})$ (resp. $U(\mathfrak{g})$) module with highest weight λ and highest weight vector v_λ. Then

$$ch(V^q(\lambda)) = ch(V(\lambda))$$

$(\Rightarrow \forall \mu \in P,\ \dim V^q(\lambda)_\mu = \dim V(\lambda)_\mu)$

and $$ch(V^q(\lambda)) = \frac{\sum_{w\in W} (-1)^{l(w)} e(w(\lambda+\rho)-(\lambda+\rho))}{\prod_{\alpha\in\Delta^+} (1-e(-\alpha))^{\dim \mathfrak{g}_\alpha}}.$$

given by the Weyl-Kac character formula.

Recall:

Thm: Let $V^q \in \mathcal{O}^q_{int}$. For each $i \in I$,

$$V^q = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} a_k V^q(k) \quad , \quad a_k \in \mathbb{Z}_{\geq 0}$$

where $V^q(k)$ are the irred. $U_q(\mathfrak{g})_i \cong U_q(sl(2))$ submodules.

Let $V^q \in \mathcal{O}^q_{int}$ highest wt irred. $U_q(\mathfrak{g})$ module with highest wt $\lambda \in P$.

$\Rightarrow \lambda \in P^+$ & $V^q \cong V^q(\lambda)$

Conversely, each irred. $U_q(\mathfrak{g})$-module in $\mathcal{O}^q_{int}$ is isomorphic to $V^q(\lambda)$ for some $\lambda \in P^+$.

<u>Thm</u>: (Complete reducibility Thm)

Every $U_q(\mathfrak{g})$-module $V^q \in \mathcal{O}^q_{int}$ is completely reducible.

$$\left(\Rightarrow V^q \cong \bigoplus_{i=1}^{r} V^q(\lambda_i)\ ,\ \lambda_i \in P^+\right)$$

<u>Cor.</u>: Tensor product of a finite number of $U_q(\mathfrak{g})$-modules in $\mathcal{O}^q_{int}$ is completely reducible.

$V^q \in \mathcal{O}^q_{int}$

$\Rightarrow V^q = \bigoplus_{\mu \in P} V^q_\mu \,, \quad \dim V^q_\mu < \infty .$

Define

$$(V^q)^* = \bigoplus_{\mu \in P} (V^q_\mu)^*$$

called the <u>restricted dual</u> of V^q

$(V^q)^*$ is a $U_q(\mathfrak{g})$-module via

$\forall \; x \in U_q(\mathfrak{g}) \,, \quad f \in (V^q)^* \,, \; v \in V^q$

$$(x.f)(v) = f(S(x).v)$$

(exer.)

Recall $\quad S: U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ antipode

$$S(e_i) = -e_i q_i^{h_i} \,, \quad S(f_i) = -q_i^{-h_i} f_i$$

$$S(q^h) = q^{-h} \quad \forall \; h \in \check{P} \,, \; i \in I .$$

Recall $\quad V^q \in \mathcal{O}^q_{int}$

$\Rightarrow \mathrm{wt}(V^q) \subset \bigcup_{j=1}^{s} D(\lambda_j)$, where

$$D(\lambda_j) = \left\{ \lambda_j - \sum_{i\in I} m_i \alpha_i \;\middle|\; m_i \in \mathbb{Z}_{\geq 0} \right\}$$

Prop: (1) $\mathrm{wt}\left(V^q\right)^* \subset \bigcup_{j=1}^{s} \left(-\lambda_j + Q_+\right)$

$\left(Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0} \alpha_i \right)$

(2) $\mu \in \mathrm{wt}(V^q)$, then

$$\left(V^q_\mu\right)^* = \left(V^q\right)^*_{-\mu}$$

Pf. (2) Choose ordered basis $\{u_1, u_2, \cdots, u_k\}$ for V^q_μ. Take the dual basis $\{g_1, g_2, \cdots, g_k\}$ for $\left(V^q_\mu\right)^*$ defined by

$$g_j(u_i) = \delta_{ij}$$

Let $u_i \in V^q_\mu \Rightarrow q^h \cdot u_i = q^{\mu(h)} u_i$

Consider the corresponding dual bases vector g_i:

$$\left(q^h \cdot g_i\right)(u_j) = g_i\left(S(q^h)\cdot u_j\right) = g_i\left(q^{-h}\cdot u_j\right)$$

$$= g_i\left(q^{-\mu(h)} u_j\right) = q^{-\mu(h)} g_i(u_j) \quad \forall j \in \{1, 2, \cdots, k\}$$

$$\Rightarrow \quad q^h \cdot g_i = q^{-\mu(h)} g_i \quad \forall h \in \check{P},\ i \in \{1, 2, \cdots, k\}$$

$$\Rightarrow \quad g_i \in (V^q)^*_{-\mu} \quad \forall \ i \in \{1, 2, \cdots, k\}$$

proving (2). Hence (1) follows (exer.).

For $\lambda \in P^+$, $V^q(\lambda) \in \mathcal{O}^q_{int}$

$$V^q(\lambda) = U_q(g) \cdot v_\lambda, \quad e_i \cdot v_\lambda = 0 \quad \forall i \in I$$

$$q^h \cdot v_\lambda = q^{\lambda(h)} v_\lambda, \quad f_i^{\lambda(h_i)+1} \cdot v_\lambda = 0 \quad \forall i \in I$$

$$\Rightarrow \quad \dim V^q(\lambda)_\lambda = 1.$$

$$V^q(\lambda)_\lambda = \operatorname{span}\{v_\lambda\}$$

$$(V^q(\lambda)_\lambda)^* = \operatorname{span}\{v_\lambda^*\}, \quad v_\lambda^*(v_\lambda) = 1,$$

and $v_\lambda^* \in (V^q(\lambda))^*$, $v_\lambda^*(u) = 0 \ \forall u \in V^q(\lambda)_\mu$, $\mu \neq \lambda$.

Consider

$$\underbrace{U_q(g) \cdot v_\lambda^*}_{\parallel \atop W^*} \subseteq (V^q(\lambda))^*$$

<u>Claim</u>: $W^* = V^q(\lambda)^*$.

(*) $q^h \cdot v_\lambda^* = q^{-\lambda(h)} v_\lambda^*$, $u_\mu \in V^q(\lambda)_\mu$ $\forall h \in P$

$$(e_i \cdot v_\lambda^*)(u_\mu) = v_\lambda^*\left(S(e_i) \cdot u_\mu\right)$$

$$= v_\lambda^*\left(-e_i q_i^{h_i} \cdot u_\mu\right) = -q_i^{\mu(h_i)} v_\lambda^*(\underbrace{e_i \cdot u_\mu}_{\in V^q(\lambda)_{\mu+\alpha_i}})$$

$$\mu + \alpha_i = \lambda - \sum_{j \in I} m_j \alpha_j + \alpha_i$$

$$= \lambda - (m_i - 1)\alpha_i - \sum_{\substack{j \in I \\ j \neq i}} m_j \alpha_j$$

$$(e_i \cdot v_\lambda^*)(u_\mu) = -q_i^{\mu(h_i)} v_\lambda^*(e_i \cdot u_\mu)$$

$$= \begin{cases} -q_i^{\mu(h_i)}, & \text{if } m_i = 1, m_j = 0, j \neq i \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow$ For $i \in I = \{1, 2, \cdots, n\}$, we have

$$e_i \cdot v_\lambda^* \in \left(V^q(\lambda)\right)^*_{-\lambda+\alpha_i} \quad \text{and} \quad e_i \cdot v_\lambda^* \neq 0$$

(*) $$(q^h \cdot v_\lambda^*)(u_\mu) = v_\lambda^*\left(S(q^h) \cdot u_\mu\right) \quad \forall u_\mu \in V^q(\lambda)_\mu$$

$$= v_\lambda^*(q^{-h} \cdot u_\mu) = v_\lambda^*(q^{-\mu(h)} u_\mu) = q^{-\mu(h)} v_\lambda^*(u_\mu) = \begin{cases} q^{-\lambda(h)}, & \mu = \lambda \\ 0, & \mu \neq \lambda \end{cases}$$

$$(f_i \cdot v_\lambda^*)(u_\mu) = v_\lambda^*\left(S(f_i) . u_\mu\right)$$

$$= v_\lambda^*\left(-q_i^{h_i} f_i \cdot u_\mu\right) = v_\lambda^*\left(-q_i^{h_i} \cdot (f_i \cdot u_\mu)\right)$$

$$= -v_\lambda^*\Big(\underbrace{q^{h_i} \cdot (f_i \cdot u_\mu)}_{\in\, V^q(\lambda)_{\mu-\alpha_i}}\Big)$$

$$\mu - \alpha_i = \lambda - \sum_{j=1}^{n} m_j \alpha_j - \alpha_i \,, \quad m_j \in \mathbb{Z}_{\geq 0}$$

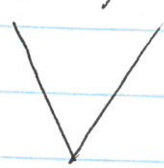
$$= \lambda - (m_i + 1)\alpha_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \alpha_j$$

$$\neq \lambda \quad \text{since} \quad m_j \in \mathbb{Z}_{\geq 0}, \; m_i \neq -1$$

$$\Rightarrow (f_i \cdot v_\lambda^*) u_\mu = -v_\lambda^*\left(q^{h_i} \cdot f_i \cdot u_\mu\right) = 0 \;\; \forall \mu \in wt(V^q(\lambda))$$

$$\Rightarrow f_i \cdot v_\lambda^* = 0 \quad \forall\, i \in I.$$

$\Rightarrow W^* \cong V^q(\lambda)^*$ whoose wts are of the for $-\lambda + \sum_{j=1}^{n} m_j \alpha_j$, $m_j \in \mathbb{Z}_{\geq 0}$



$-\lambda$

and is called the lowest wt. module with lowest wt. $-\lambda$ and lowest wt. vector v_λ^*.