$A = (a_{ij})_{n\times n}$ symmetrizable GCM

$\mathfrak{g} = \mathfrak{g}(A)$ Kac-Moody Lie alg.

$U_q(\mathfrak{g})$ = associated quantum group.

Verma module: $\lambda \in P$

$$M^q(\lambda) = U_q(\mathfrak{g}) / J^q(\lambda)$$

$J^q(\lambda)$ is the ideal of $U_q(\mathfrak{g})$ gen. by
$\{e_i, q^h - q^{\lambda(h)}1 : i \in I, h \in \check{P}\}$.

$M^q(\lambda)$ is a highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$.

Properties of $M^q(\lambda)$:

(1) As a $U_q^-(\mathfrak{g}) := U_q^-$-module, $M^q(\lambda)$ is free of rank 1.

(Recall $M^q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda = (U_q^- \otimes U_q^0 \otimes U_q^+) \cdot v_\lambda$
$\cong U_q^-(\mathfrak{g}) \cdot v_\lambda$ as vector space.)

(2) Let $V^q(\lambda)$ be any highest weight $U_q(\mathfrak{g})$ module with highest $\lambda \in P$ and highest weight vector u_λ. Then $\exists$ a homomorphism

$$\begin{aligned} \varphi : M^q(\lambda) &\longrightarrow V^q(\lambda) \\ v_\lambda = 1 + J^q(\lambda) &\longmapsto u_\lambda \end{aligned}$$

which is onto.

(3) $M^q(\lambda)$ has a unique maximal submodule $N^q(\lambda)$. So

$$V^q(\lambda) = M^q(\lambda) / N^q(\lambda)$$

is an irred. highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector

$$u_\lambda = v_\lambda + N^q(\lambda).$$

Integrable $U_q(\mathfrak{g})$-module:

An $U_q(\mathfrak{g})$-module $V^q \in \mathcal{O}^q$ is integrable if $e_i, f_i, i \in I$ act locally nilpotently on V^q.

Category $\mathcal{O}^q_{int}$:

Objects : set of integrable $U_q(\mathfrak{g})$ modules in $\mathcal{O}^q$.

(So for $V^q \in \mathcal{O}^q_{int}$ we have

(1) $V^q = \bigoplus_{\mu \in P} V^q_\mu$, $\dim V^q_\mu < \infty$

(2) $wt(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_k)$ for finitely many $\lambda_1, \cdots, \lambda_k \in P$

(3) e_i, f_i, $i \in I$ act locally nilpotently on V^q.)

Morphisms ; $U_q(\mathfrak{g})$-module homomorphisms.

Fact : (1) $\mathcal{O}^q_{int}$ is closed under direct sum:

($V_i^q \in \mathcal{O}^q_{int}$, $i \in J$ then $\bigoplus_{i \in J} V_i^q \in \mathcal{O}^q_{int}$)

(2) $\mathcal{O}^q_{int}$ is closed under finite tensor products:

($V_i^q \in \mathcal{O}^q_{int}$, $i = 1, 2, \cdots, k$, then $\bigotimes_{i=1}^{k} V_i^q \in \mathcal{O}^q_{int}$.

Exer: For each $i \in I$, $k \in \mathbb{Z}_{>0}$ we have

$$e_i f_i^{(k)} = f_i^{(k)} e_i + f_i^{(k-1)} \frac{q_i^{(h_i-k+1)} - q_i^{-(h_i-k+1)}}{q_i - q_i^{-1}}$$

(Recall $f_i^{(k)} = \frac{1}{[k]_{q_i}} f^k$, $q_i = q^{s_i h_i}$)

Fix $i \in I$. Define

$$\begin{aligned} U_q(\mathfrak{g})_{(i)} &= \text{subalg. of } U_q(\mathfrak{g}) \text{ gen. by } \\ &\quad \{e_i, f_i, q_i^{\pm h_i}\} \\ &\cong U_{q_q}(\mathfrak{sl}(2)) \end{aligned}$$

Thm: Let $V^q \in \mathcal{O}^q_{int}$. For each $i \in I$, V^q decomposes into direct sum of $U_q(\mathfrak{h})$-invariant finite dim'l irred. $U_q(\mathfrak{g})_{(i)}$-submodules.

Thm: Let $\lambda \in P^+$ and $V^q(\lambda)$ be the irred. highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. Then

$$f_i^{\lambda(h_i)+1} \cdot v_\lambda = 0 \quad \forall i \in I.$$

$(\Rightarrow \; N^q(\lambda) = \langle f_i^{\lambda(h_i)+1} \cdot v_\lambda \mid i \in I \rangle$.

<u>Proof</u>: Recall, for $i \in I$, $k \in \mathbb{Z}_{>0}$

$$e_i f_i^{(k)} \cdot v_\lambda = \underbrace{f_i^{(k)} e_i \cdot v_\lambda}_{\overset{\shortparallel}{0}} + f_i^{(k-1)} \frac{q_i^{h_i-k+1} - q_i^{-h_i+k-1}}{q_i - q_i^{-1}} \cdot v_\lambda$$

$$= f_i^{(k-1)} \frac{q_i^{h_i-k+1} \cdot v_\lambda - q_i^{-h_i+k-1} \cdot v_\lambda}{q_i - q_i^{-1}}$$

$$= f_i^{(k-1)} \left(\frac{q_i^{\lambda(h_i)-k+1} - q_i^{-\lambda(h_i)+k-1}}{q_i - q_i^{-1}} \right) \cdot v_\lambda$$

$$= [\lambda(h_i) - k + 1]_{q_i} f_i^{(k-1)} \cdot v_\lambda$$

Setting $k = \lambda(h_i) + 1$, we have

$$e_i f_i^{(\lambda(h_i)+1)} \cdot v_\lambda = [0]_{q_i} f_i^{(\lambda(h_i))} \cdot v_\lambda$$

$$= 0 f_i^{(\lambda(h_i))} \cdot v_\lambda = 0$$

Since $[0]_{q_i} = \frac{q_i^0 - q_i^0}{q_i - q_i^{-1}} = 0$.

For $i \neq j$,

$$e_i f_j^{(\lambda(h_i)-k+1)} \cdot v_\lambda = f_j^{(\lambda(h_i)-k+1)} e_i \cdot v_\lambda = 0$$

since $e_i f_j = f_j e_i$ for $i \neq j$.

For $h \in \check{P}$,

$$q^h f_i^{(\lambda(h_i)+1)} \cdot v_\lambda = q^{-\alpha_i(h)(\lambda(h_i)+1)} f_i^{(\lambda(h_i)+1)} q^h \cdot v_\lambda$$

since

$$\left(q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \Rightarrow q^h f_i = q^{-\alpha_i(h)} f_i q^h \right)$$

$$= q^{\lambda(h) - \alpha_i(h)(\lambda(h_i)+1)} f_i^{(\lambda(h_i)+1)} \cdot v_\lambda$$

Suppose for some $i \in I$,

$$f_i^{\lambda(h_i)+1} \cdot v_\lambda \neq 0$$

$\Rightarrow \quad \langle f_i^{\lambda(h_i)+1} \cdot v_\lambda \rangle \neq 0$ is a submodule of $V^q(\lambda)$ which is a contradiction since $V^q(\lambda)$ is irreducible. //