$A = (a_{ij})_{n \times n}$ symmetrizable GCM

$D = \mathrm{diag}(s_1, \dots, s_n)$, DA symmetric.
$s_i \in \mathbb{Z}_{>0}$. $I = \{1, 2, \dots, n\}$

$(A, \Pi, \check{\Pi}, P, \check{P})$ Cartan Datum

$\Pi = \{\alpha_1, \dots, \alpha_n\}$ simple roots

$\check{\Pi} = \{h_1, \dots, h_n\}$ simple coroots

$\check{P} = \bigoplus_{i=1}^{n} \mathbb{Z} h_i \oplus \bigoplus_{s=1}^{k} \mathbb{Z} d_s$, $k = \mathrm{corank}(A)$.

$\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$ Cartan subalg. of $\mathfrak{g} = \mathfrak{g}(A)$

$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(\check{P}) \subset \mathbb{Z}\}$.

$U_q(\mathfrak{g}) = \langle \{e_i, f_i \mid i \in I\} \cup \{q^h \mid h \in \check{P}\} \rangle$

is an assoc. alg. over $\mathbb{C}(q)$ with unity, satisfying following relations:

1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ $\forall h, h' \in \check{P}$

2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ $\forall i \in I, h \in \check{P}$

3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ $\forall i \in I, h \in \check{P}$

4) $e_i f_i - f_i e_i = \delta_{ij} \cdot \dfrac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$, $q_i = q^{s_i}$

5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \;\forall i \neq j$

6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \;\forall i \neq j$

$\mathcal{U}_q = \mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra with the

~~counit~~ comultiplication $\Delta : \mathcal{U}_q \longrightarrow \mathcal{U}_q \otimes \mathcal{U}_q$

counit $\varepsilon : \mathcal{U}_q \longrightarrow \mathbb{C}(q)$

and antipode $S : \mathcal{U}_q \longrightarrow \mathcal{U}_q$ given by:

$$\Delta(q^h) = q^h \otimes q^h \quad \forall\, h \in \check{P}$$

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad i \in I$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \quad i \in I$$

where $t_i = q_i^{h_i} = q^{s_i h_i}$ and Δ is an algebra homomorphism.

$$\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = 0 = \varepsilon(f_i) \quad \forall\, i \in I$$

and ε is an alg. hom.

$$S(q^h) = q^{-h}, \quad S(e_i) = -e_i t_i, \quad S(f_i) = -t_i^{-1} f_i$$

and S is an antihomomorphism.

$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \cdots \oplus \mathbb{Z}\alpha_n$

$Q^+ = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \mathbb{Z}_{\geq 0}\alpha_2 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_n$

$Q^- = -Q^+$

For $\mu, \nu \in \mathfrak{h}^*$, we say

$\mu \geq \nu$ if $\mu - \nu \in Q^+$.

For $\alpha \in Q$, define

$$U_q(\mathfrak{g})_\alpha = \{u \in U_q \mid q^h u q^{-h} = q^{\alpha(h)} u \ \forall h \in P^\vee\}$$

Then $U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha$
(rootspace decomp.)

$U_q^+(\mathfrak{g})$ = subalg. of $U_q(\mathfrak{g})$ generated by $\{e_i \mid i \in I\}$

$U_q^-(\mathfrak{g})$ = subalg. of $U_q(\mathfrak{g})$ generated by $\{f_i \mid i \in I\}$

$U_q^0(\mathfrak{g})$ = subalg. of $U_q(\mathfrak{g})$ generated by $\{q^h \mid h \in P^\vee\}$

Then $U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})$
(triangular decomp.)

Define linear map $T: \mathcal{U}_q(\mathfrak{g}) \longrightarrow \mathcal{U}_q(\mathfrak{g})$

$$T(e_i) = f_i \,,\quad T(f_i) = e_i \,, T(q^h) = q^{-h}$$

$\forall\, i \in I,\ h \in \check{P}$.

T is an alg. hom., hence automorphism with $T^2 = id$.

Note: $T(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^-(\mathfrak{g})$.

Recall that we are assuming

$$q^m \neq 1 \quad \text{for any } m \in \mathbb{Z}_{>0}.$$

(sometimes, this is referred to as 'q' generic)

<u>$\mathcal{U}_q(\mathfrak{g})$ representation theory:</u>

1) Weight module:

A $\mathcal{U}_q(\mathfrak{g})$-module V^q is a weight module if

$$V^q = \bigoplus_{\mu \in P} V^q_\mu$$

where

$$V^q_\mu = \{ v \in V^q \mid q^h \cdot v = q^{\mu(h)} v \ \forall\, h \in \check{P} \}$$

If $V^q_\mu \neq 0$, then μ is a <u>weight</u> of V_q,

and we call V^q_μ to be the μ-weight space and $0 \neq u \in V^q_\mu$ to be a μ-weight vector.
A weight vector $0 \neq u \in V^q_\mu$ is a highest weight vector if

$$e_i \cdot u = 0 \quad \forall i \in I.$$

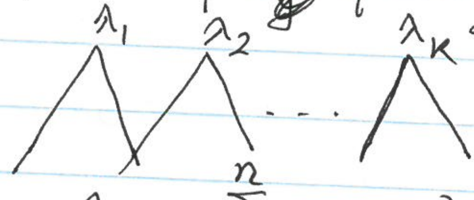
$wt(V^q_\mu)$ = set of all weights of V^q.

<u>Category $\mathcal{O}^q$:</u>

Objects: $U_q(g)$ weight modules V^q satisfying

- $\dim V^q_\mu < \infty \quad \forall \mu \in wt(V^q)$
- $wt(V^q_\mu) \subseteq D(\lambda_1) \cup D(\lambda_2) \cup \cdots \cup D(\lambda_k)$
  finitely many $\lambda_1, \lambda_2, \cdots, \lambda_k \in P$.

(Recall $D(\lambda_j) = \{\mu \in P \mid \mu - \lambda_j \in Q^-\}$

so $wt(V^q_\mu) \subset$ )

$\left(\mu \in D(\lambda_j) \Rightarrow \mu = \lambda_j - \sum_{i=1}^{n} m_i \alpha_i \right)$, $m_i \in \mathbb{Z}_{\geq 0}$

Morphisms: $U_q(g)$-module homomorphisms.

For $V^q \in \mathcal{O}^q$ we define character as:

$$\operatorname{ch} V^q = \sum_{\mu \in wt(V^q)} (\dim V^q_\mu)\, e(\mu)$$

if $V^q = \bigoplus_{\mu \in wt(V^q)} V^q_\mu$.

A highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ with highest weight λ is a weight module such that $\exists\, 0 \neq v_\lambda \in V^q(\lambda)$ with

1) $V^q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$
2) $e_i \cdot v_\lambda = 0 \quad \forall\, i \in I$
3) $h \cdot v_\lambda = q^{\lambda(h)} v_\lambda \quad \forall\, h \in \check{P}$

Hence $V^q(\lambda) = U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) \cdot v_\lambda$

$$\cong U_q^-(\mathfrak{g}) \cdot v_\lambda \quad \text{(as vect. spaces)}$$

$\Longrightarrow$ $u \in V^q(\lambda)$ then

$$u = f_{i_1} \cdots f_{i_k} \cdot v_\lambda$$

$\Longrightarrow$ $\dim V^q(\lambda)_\lambda = 1$

Of course, $V^q(\lambda) \in \mathcal{O}^q$.

For $\lambda \in P$

Define $J^q(\lambda)$ = left ideal of $U_q(\mathfrak{g})$ gen. by $\{e_i, q^h - q^{\lambda(h)} 1 \mid i \in I, h \in \check{P}\}$

Then
$$M^q(\lambda) = U_q(\mathfrak{g}) / J^q(\lambda)$$
is a left $U_q(\mathfrak{g})$-module called the quantum Verma module which is gen. by
$$v_\lambda = 1 + J^q(\lambda) \in M^q(\lambda)$$

Note
$$e_i \cdot v_\lambda = 0 \quad \forall i \in I$$
$$q^h \cdot (1 + J^q(\lambda)) = q^h \cdot v_\lambda = q^{\lambda(h)} v_\lambda \quad \forall h \in \check{P}$$

$\Rightarrow$ $M^q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$.