$\mathcal{O}_{int}$ = category of integrable $\mathfrak{g}$-modules.

For $\lambda \in P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}\}$

$\exists!$ (up to isom.) irreducible highest weight $\mathfrak{g}$-module $V(\lambda)$ with highest wt. λ and highest weight vector v_λ.

- $V(\lambda) = U(\mathfrak{g}).v_\lambda$
- $e_i . v_\lambda = 0 \quad \forall i \in I$
- $h . v_\lambda = \lambda(h) v_\lambda \quad \forall h \in \mathfrak{h}$
- $f_i^{\lambda(h_i)+1} . v_\lambda = 0 \quad \forall i \in I$

Recall

$$V(\lambda) = M(\lambda)/N(\lambda) \in \mathcal{O}_{int}$$

<u>Thm</u> (1) Every $\mathfrak{g}$-module $V \in \mathcal{O}_{int}$ is completely reducible.

(i.e. $V = \bigoplus_j V(\lambda_j)$, $\lambda_j \in P^+$.)

(2) $V_1, V_2, \cdots, V_k \in \mathcal{O}_{int}$

$\Rightarrow V_1 \otimes V_2 \otimes \cdots \otimes V_k \in \mathcal{O}_{int}$

(hence completely reducible.)

For $V \in \mathcal{O}_{int}$, $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$

$\dim V_\mu < \infty$ for each $\mu \in \mathfrak{h}^*$

Hence

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu)\, e(\mu)$$

Thm:

Weyl-Kac Character formula: For $\lambda \in P^+$

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w(\lambda+\rho)-\rho)}{\prod_{\alpha \in \Delta_+} (1-e(-\alpha))^{\dim \mathfrak{g}_\alpha}}.$$

Cor: Setting $\lambda = 0$, we have

$$\prod_{\alpha \in \Delta_+} (1-e(-\alpha))^{\dim \mathfrak{g}_\alpha} = \sum_{w \in W} (-1)^{\ell(w)} e(w\rho - \rho)$$

called the "Denominator Formula".

Prop: $w(\lambda+\rho)-\rho = -\sum_{i=1}^{n} m_i \alpha_i$, $m_i \in \mathbb{Z}_{\geq 0}$

$$\Rightarrow e(w(\lambda+\rho)-\rho) = e\left(\sum_{i=1}^{n} m_i(-\alpha_i)\right)$$

$$= \prod_{i=1}^{n} e(m_i(-\alpha_i)) = \prod_{i=1}^{n} e(-\alpha_i)^{m_i}.$$

Ex (1) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{sl}(3, \mathbb{C})$

$$W = S_3 = \{1, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$$

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\},$$

$\dim(\mathfrak{g}_\alpha) = 1 \quad \forall\, \alpha \in \Delta_+$

Denominator formula:

$$\prod_{\alpha \in \Delta_+} (1 - e(-\alpha)) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} e(\sigma\rho - \rho) \qquad (1)$$

$$\rho = \frac{1}{2}\left(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)\right) = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$$

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3 .$$

$$\mathrm{LHS} = (1 - e(-\alpha_1))(1 - e(-\alpha_2))(1 - e(-\alpha_1 - \alpha_2))$$

RHS = ?

$w = (1)$: $w\rho - \rho = 0$

$w = (1\,2)$: $w\rho - \rho = (1\,2)(\varepsilon_1 - \varepsilon_3) - (\varepsilon_1 - \varepsilon_3)$

$$= \varepsilon_2 - \varepsilon_3 - \varepsilon_1 + \varepsilon_3 = -(\varepsilon_1 - \varepsilon_2) = -\alpha_1$$

$w = (2\,3)$: $w\rho - \rho = (2\,3)(\varepsilon_1 - \varepsilon_3) - (\varepsilon_1 - \varepsilon_3)$

$$= \varepsilon_1 - \varepsilon_2 - \varepsilon_1 + \varepsilon_3 = -(\varepsilon_2 - \varepsilon_3) = -\alpha_2$$

$$w=(13):\quad w\rho-\rho=(13)(\varepsilon_1-\varepsilon_3)-(\varepsilon_1-\varepsilon_3)$$
$$=\varepsilon_3-\varepsilon_1-\varepsilon_1+\varepsilon_3=-2(\varepsilon_1-\varepsilon_3)$$
$$=-2\alpha_1-2\alpha_2$$

$$w=(123):\quad w\rho-\rho=(123)(\varepsilon_1-\varepsilon_3)-(\varepsilon_1-\varepsilon_3)$$
$$=(\varepsilon_2-\varepsilon_1)-\varepsilon_1+\varepsilon_3=-(\varepsilon_1-\varepsilon_2)-(\varepsilon_1-\varepsilon_3)$$
$$=-2\alpha_1-\alpha_2$$

$$w=(132):\quad w\rho-\rho=(132)(\varepsilon_1-\varepsilon_3)-(\varepsilon_1-\varepsilon_3)$$
$$=\varepsilon_3-\varepsilon_2-\varepsilon_1+\varepsilon_3=-(\varepsilon_2-\varepsilon_3)-(\varepsilon_1-\varepsilon_3)$$
$$=-\alpha_1-2\alpha_2$$

$\Rightarrow$ RHS of (1) is

$$e(0)-e(-\alpha_1)-e(-\alpha_2)-e(-2\alpha_1-2\alpha_2)$$
$$+e(-2\alpha_1-\alpha_2)+e(-\alpha_1-2\alpha_2)$$

$$=1-e(-\alpha_1)-e(-\alpha_2)-e(-2\alpha_1-2\alpha_2)$$
$$+e(-2\alpha_1-\alpha_2)+e(-\alpha_1-2\alpha_2)$$
$$=(1-e(-\alpha_1))-e(-\alpha_2)(1-e(-2\alpha_1))$$
$$+e(-\alpha_1-2\alpha_2)(1-e(-\alpha_1))$$
$$=(1-e(-\alpha_1))\left(1-e(-\alpha_2)(1+e(-\alpha_1))+e(-\alpha_1-2\alpha_2)\right)$$

$$= (1-e(-\alpha_1))\big((1-e(-\alpha_2)) - e(-\alpha_1-\alpha_2)(1-e(-\alpha_2))\big)$$

$$= (1-e(-\alpha_1))(1-e(-\alpha_2))(1-e(-\alpha_1-\alpha_2))$$

$$= \text{LHS}.$$

<u>Ex(2)</u> $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$$\mathfrak{g} = \widehat{sl(2,\mathbb{C})} = sl(2,\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where $[x\otimes t^i, y\otimes t^j] = [x,y]\otimes t^{i+j} + \delta_{i+j,0}\, tr(xy)\, i c$

$[c, \mathfrak{g}] = 0$

$[d, x\otimes t^i] = i(x\otimes t^i)$ $\left(d = 1\otimes t\frac{d}{dt}\right)$

$[d, d] = 0.$

called an affine Lie alg. associated to the affine GCM A.

Simple roots : α_1, α_2

Null root : $\delta = \alpha_1 + \alpha_2$

$$\Delta_+ = \{\alpha_1 + k\delta, \alpha_2 + k\delta \mid k \geq 0\} \cup \{k\delta \mid k > 0\}$$

$$\Delta = \Delta_+ \cup \Delta_- \quad , \Delta_- = -\Delta_+ .$$

Weyl group

$$W = \langle r_1, r_2 \rangle = \{ (r_1 r_2)^m, r_1 (r_1 r_2)^m \mid m \in \mathbb{Z} \}$$

is an infinite dihedral group.

$\rho \in \mathfrak{h}^*$, $\rho(h_1) = 1$, $\rho(h_2) = 1$.

Denominator formula:

$$\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\dim \mathfrak{g}_\alpha} = \sum_{w \in W} (-1)^{\ell(w)} e(w\rho - \rho) \quad (2)$$

LHS of (2): $(\dim \mathfrak{g}_\alpha = 1 \ \forall \alpha \in \Delta_+)$

$$\prod_{k \geq 0} (1 - e(-\alpha_1 - k\delta))(1 - e(-\alpha_2 - k\delta)) \prod_{k > 0} (1 - e(k\delta))$$

Set $e(-\alpha_1) = p$, $e(-\alpha_2) = q \Rightarrow e(-\delta) = e(-\alpha_1 - \alpha_2) = pq$

$$= \prod_{k=0}^{\infty} (1 - p(pq)^k)(1 - q(pq)^k) \prod_{k=1}^{\infty} (1 - (pq)^k)$$

$$= \prod_{k=1}^{\infty} (1 - p(pq)^{k-1})(1 - q(pq)^{k-1})(1 - (pq)^k)$$

$$= \prod_{k=1}^{\infty} (1 - p^k q^{k-1})(1 - p^{k-1} q^k)(1 - p^k q^k) \qquad (*)$$

(*) is called "Jacobi triple product".

Jacobi Triple Product identity:

$$\prod_{k=1}^{\infty}(1-p^k q^{k-1})(1-p^{k-1}q^k)(1-p^k q^k)$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k p^{\binom{k}{2}} q^{\binom{k+1}{2}}$$

<u>RHS:</u> $\sum_{w\in W} (-1)^{\ell(w)} e(w\rho-\rho)$

$w=1$: $w\rho-\rho=0$

$w=r_1$: $w\rho-\rho = r_1\rho-\rho = \rho-\rho(h_1)\alpha_1-\rho = -\alpha_1$

$w=r_2r_1$: $w\rho-\rho = r_2r_1\rho-\rho = r_2(\rho-\alpha_1)-\rho$

$= (\rho-\alpha_1)-(\rho-\alpha_1)(h_2)\alpha_2-\rho$

$= \rho-\alpha_1-(1+2)\alpha_2-\rho = -\alpha_1-3\alpha_2$

$e(r_2r_1-\rho) = e(-\alpha_1-3\alpha_2) = pq^3$

Rest left as exercise.