$\lambda \in \mathfrak{h}^*$

$V(\lambda)$ ~~to~~ be a highest weight $\mathfrak{g}$-module.

$\Rightarrow$

1) $V(\lambda)$ is a weight module

2) $\exists\ 0 \neq v_\lambda \in V(\lambda)$ such that

$$\mathcal{U}(\mathfrak{g}) \cdot v_\lambda = V(\lambda)$$

3) $e_i \cdot v_\lambda = 0 \quad \forall\, i \in I$

$$h \cdot v_\lambda = \lambda(h) v_\lambda \quad \forall\, h \in \mathfrak{h}.$$

Recall

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}^-(\mathfrak{g}^\bullet) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}^+(\mathfrak{g}^\bullet)$$

$\mathcal{U}^\pm(\mathfrak{g}^\bullet)$ = associative alg. gen. by $\begin{cases} e_i, & i \in I \\ f_i, & i \in I \end{cases}$

$\mathcal{U}(\mathfrak{h})$ = assoc. alg. gen. by $\check{P}$.

$$\Rightarrow V(\lambda) = \mathcal{U}(\mathfrak{g}) \cdot v_\lambda$$

$$= \mathcal{U}^-(\mathfrak{g}^\bullet) \underbrace{\mathcal{U}(\mathfrak{h}) \mathcal{U}^+(\mathfrak{g}^\bullet) \cdot v_\lambda}_{\text{scalar mult. of } v_\lambda}$$

$$\cong \mathcal{U}^-(\mathfrak{g}^\bullet) \cdot v_\lambda \quad \text{(as vect. spaces)}$$

$\Rightarrow v \in V(\lambda)$, then

$$v = f_{i_1} f_{i_2} \cdots f_{i_k} \cdot v_\lambda$$

$$\Rightarrow \quad wt(v) = \lambda - \alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_k}$$

Hence we have:

1) $wt(V(\lambda)) = \{\lambda - \alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_k} \mid k \geq 0, i_j \in I\}$
$\subseteq D(\lambda)$

2) $\dim V(\lambda)_\lambda = 1$

3) $\dim V(\lambda)_\mu < \infty \quad \forall\ \mu \in wt(V(\lambda))$

Fix $\lambda \in \mathfrak{h}^*$. Define

$J(\lambda)$ = ideal of $\mathcal{U}(\mathfrak{g})$ gen. by
$\{e_i \mid i \in I\} \cup \{h - \lambda(h) \mid h \in \mathfrak{h}\}$

Denote $\quad M(\lambda) = \mathcal{U}(\mathfrak{g}) / J(\lambda)$

$M(\lambda)$ is an $\mathcal{U}(\mathfrak{g})$-module (hence $\mathfrak{g}$-module) via:

$$x \cdot (y + J(\lambda)) := (xy) + J(\lambda)$$

$\forall\ x, y \in \mathcal{U}(\mathfrak{g})$.

$M(\lambda)$ is called the Verma module.

Properties of $M(\lambda)$:

(1) $M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$ $(\Rightarrow M(\lambda) \in \mathcal{O})$

(2) If $V(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight vector v_λ, then $\exists$ a $\mathfrak{g}$-module homomorphism

$$\Phi : M(\lambda) \twoheadrightarrow V(\lambda) \quad (\text{onto})$$
$$1 + J(\lambda) \longmapsto v_\lambda$$

(3) As a $\mathcal{U}^-(\mathfrak{g})$-module $M(\lambda)$ is free of rank 1 gen. by $v_\lambda = 1 + J(\lambda)$.
$(\Rightarrow M(\lambda) \cong \mathcal{U}^-(\mathfrak{g}) \cdot (1 + J(\lambda)))$

(4) $M(\lambda)$ has a unique max'l submodule denote by $N(\lambda)$.
$(\Rightarrow M(\lambda)/N(\lambda)$ is irreducible $\mathfrak{g}$-module,
$M(\lambda)/N(\lambda) \in \mathcal{O}$

<u>Thm:</u> Every irreducible $\mathfrak{g}$-module in $\mathcal{O}$ is isomorphic to $V(\lambda) = M(\lambda)/N(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

<u>Defn:</u> A weight module V of $\mathfrak{g}$ is called <u>integrable</u> (or standard) if e_i and f_i, $i \in I$ act locally nilpotently:

i.e. For each $v \in V$, $i \in I$ $\exists N = N(v,i) \in \mathbb{N}$ such that $e_i^N \cdot v = 0$ & $f_i^N \cdot v = 0$.

<u>Category $\mathcal{O}_{int}$:</u>

- Objects: $V \in \mathcal{O}_{int}$ if
  (1) $V \in \mathcal{O}$
  (2) V is integrable $\mathfrak{g}$-module.
- Homomorphisms: $\mathfrak{g}$-module homomorphisms.

So for $V \in \mathcal{O}_{int} \Rightarrow$

(1) $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, $\dim V_\mu < \infty$

(2) $wt(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_k)$

$\lambda_1, \dots, \lambda_k \in \mathfrak{h}^*$

$$D(\lambda_j) = \{ \lambda_j - \sum_{i \in I} m_i \alpha_i \mid m_i \in \mathbb{Z}_{\geq 0} \}$$

(3) For each $v \in V$ $\exists$ $N \in \mathbb{N}$ such that

$$e_i^N \cdot v = 0, \quad f_i^N \cdot v = 0, \quad i \in I$$

For $i \in I$, we denote

$\mathfrak{g}_i$ = subalg. of $\mathfrak{g}$ generated by $\{e_i, f_i, h_i = [e_i, f_i]\}$

$\cong \mathfrak{sl}(2)$.

$\mathcal{U}_i$ = subalg. of $\mathcal{U}(\mathfrak{g})$ gen. by $\{e_i, f_i, h_i\}$

$\cong \mathcal{U}(\mathfrak{sl}(2))$.

Let $V \in \mathcal{O}_{int}$.

Since e_i, f_i act locally nilpotently on V, the map

$$\tau_i = (\exp f_i)(\exp(-e_i))(\exp f_i) : V \longrightarrow V$$

is a well defined $\mathfrak{g}$-module automorphism.

<u>Thm</u>: Let $V \in \mathcal{O}_{int}$. Then we have:

(1) For each $i \in I$, V decomposes into a direct sum of finite dimensional irreducible, $\mathfrak{h}$-invariant $\mathfrak{g}_i$-submodules.

(2) For $\lambda \in wt(V)$,

$$\tau_i(V_\lambda) = V_{r_i \lambda} \quad \text{for all } i \in I$$

(Recall $r_i : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$, $r_i(\mu) = \mu - \mu(h_i)\alpha_i$ is a simple reflection.)

$\Rightarrow \dim(V_\lambda) = \dim(V_{r_i \lambda}) \ \forall i \in I$

$\Rightarrow \forall w \in W, \ \dim(V_\lambda) = \dim(V_{w\lambda})$.

Set $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \forall i \in I\}$

called the set of dominant weights.

Recall $\forall \mu \in \mathfrak{h}^*$, the highest wt. $\mathfrak{g}$-module

$$V(\mu) \cong M(\mu)/N(\mu) \in \mathcal{O}$$

<u>Thm</u>: (1) $V(\lambda) \in \mathcal{O}_{int} \Leftrightarrow \lambda \in P^+$

(2) $V \in \mathcal{O}_{int}$ irreducible

$\Rightarrow V \cong V(\lambda)$ for some $\lambda \in P^+$.

<u>Prop</u>: For $\lambda \in P^+$, the max'l submodule $N(\lambda)$ of the Verma module $M(\lambda)$ is

$$N(\lambda) = \left\langle f_i^{\lambda(h_i)+1} \cdot v_\lambda \;\middle|\; i \in I \right\rangle$$

where $v_\lambda = 1 + J(\lambda) \in M(\lambda)$.

<u>Cor</u>: For $\lambda \in P^+$, $V(\lambda) \in \mathcal{O}_{int}$

$V(\lambda) = U(\mathfrak{g}) \cdot v_\lambda$ and

$$f_i^{\lambda(h_i)+1} \cdot v_\lambda = 0, \quad \forall i \in I.$$