$A = (a_{ij})_{n\times n}$ symmetrizable GCM

$\mathfrak{g} = \mathfrak{g}(A)$ Kac-Moody Lie alg.

$(\ |\ ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ nondeg. symmetric invariant bilinear form.

Casimir element:

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u_i u^i + 2 \sum_{\alpha\in\Delta_+} \sum_i f_\alpha^{(i)} e_\alpha^{(i)}$$

Define a $\mathfrak{g}$-module V to be restricted if for each $v \in V$, $\mathfrak{g}_\alpha \cdot v = 0$ for all but finitely many positive roots α.

Note: Ω is a well defined operator on any restricted $\mathfrak{g}$-module.

Thm: (1) Let V be a restricted $\mathfrak{g}$-module. Then Ω commutes with the module action of $\mathfrak{g}$.

($\Omega : V \longrightarrow V$ & $\forall\, v \in V$, $x \in \mathfrak{g}$, $x \cdot \Omega(v) = \Omega(x \cdot v)$. Hence Ω is a $\mathfrak{g}$-module homomorphism on V.)

(2) If $v \in V$ is a maximal weight vector of weight λ (i.e. $e_i . v = 0 \ \forall i \in I$, $h.v = \lambda(h) v$ $\forall h \in \mathfrak{h}$), then

$$\Omega(v) = (\lambda + 2\rho \mid \lambda) v,$$

<u>Ex(1)</u> $A = (2)$, $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $V = \mathbb{C}^2 = V(\Lambda_1)$

$$\mathbb{C}^2 = \operatorname{span}\{v_1, v_2\}$$

$\Delta = \{\alpha, -\alpha\}$, $\Delta_+ = \{\alpha\}$, $\alpha(h) = 2$

$\rho = \frac{1}{2}\alpha = \Lambda_1$

$\varphi : \mathfrak{g} \longrightarrow \operatorname{End} V$, $\varphi = id$.

$$(x \mid y) = \underset{B_\varphi(x,y)}{\underset{\parallel}{tr(xy)}} \quad \forall \ x, y \in \mathfrak{g}$$

$$\Omega(v_1) = (\Lambda_1 + 2\rho \mid \Lambda_1)(v_1)$$
$$= \left(\tfrac{3}{2}\alpha \,\middle|\, \tfrac{1}{2}\alpha\right)(v_1) = \frac{3}{4}(\alpha \mid \alpha)(v_1) = \frac{3}{2} v_1$$

$$\Omega(v_2) = \Omega(f.v_1) = f.\Omega(v_1) = f.\left(\tfrac{3}{2} v_1\right)$$
$$= \frac{3}{2} f.v_1 = \frac{3}{2} v_2$$

$$\Rightarrow \Omega(v) = \frac{3}{2}(v) \quad \forall \ v \in V$$

$$\Rightarrow \Omega = \frac{3}{2} I. \quad //$$

<u>Ex (2)</u> $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\mathfrak{g} = sl(3, \mathbb{C})$

$\mathfrak{g}$ is a $\mathfrak{g}$-module under adjoint action.

$\mathfrak{g} = V(\lambda)$, $\lambda = \alpha_1 + \alpha_2$, $\rho = \frac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2))$
$= \alpha_1 + \alpha_2$

$\Lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, $\Lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$

$\Rightarrow \alpha_1 + \alpha_2 = \Lambda_1 + \Lambda_2 = \lambda$

$(x \mid y) = tr(xy) \quad \forall\, x, y \in \mathfrak{g}$.

$V(\lambda) = V(\Lambda_1 + \Lambda_2)$

v_λ = maximal weight vector with weight vector with weight λ.

$\Omega(v_\lambda) = (\lambda + 2\rho \mid \lambda) v_\lambda$

$= ((\alpha_1 + \alpha_2 + 2\alpha_1 + 2\alpha_2) \mid \alpha_1 + \alpha_2) v_\lambda$

$= 3(\alpha_1 + \alpha_2 \mid \alpha_1 + \alpha_2) v_\lambda$

$= 3(2 - 1 - 1 + 2) v_\lambda = 6 v_\lambda$

As before, since Ω commutes with the $\mathfrak{g}$-module action, we have

$\Omega = 6I$

<u>Exer:</u> $\mathfrak{g} = sl(3, \mathbb{C})$, $V = \mathbb{C}^3 = V(\Lambda_1)$

$\Omega = ?$

<u>Defn</u>: A $\mathfrak{g}$-module V is a weight module if it admits a weight space decomposition (i.e. $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, $V_\mu = \{v \in V \mid h.v = \mu(h)v \ \forall h \in \mathfrak{h}\}$

(1) any vector $v \in V_\mu \neq 0$ is called a weight vector of weight μ.

(2) if $e_i \cdot v = 0 \ \forall i \in I$, then v called a maximal weight vector.

(3) $\mu \in \mathfrak{h}^*$ is a weight if the <u>weight space</u> $V_\mu \neq 0$. In such case, $\dim V_\mu$ is called the multiplicity of μ.

We denote $wt(V)$ = set of weights of V.

<u>Defn</u>: If V is a weight module of $\mathfrak{g}$ with $\dim V_\mu < \infty \ \forall \mu \in wt(V)$, then we define

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu)\, e(\mu) \in \mathbb{C}[\mathfrak{h}^*]$$

with multiplication in $\mathbb{C}[\mathfrak{h}^*]$ given by

$$e(\mu)e(\nu) = e(\mu+\nu)$$

Exer: Any submodule W of a $\mathfrak{g}$-weight module V is a weight module.
(Hint: $W = \bigoplus_{\mu \in \mathfrak{h}^*} W_\mu$, $W_\mu = W \cap V_\mu$.)

Category $\mathcal{O}$:

Objects: $\mathfrak{g}$-modules V satisfying:

(1) V is a weight module with $\dim V_\mu < \infty \quad \forall \mu \in wt(V)$.

(2) $\exists$ finitely many weights $\lambda_1, \lambda_2, \cdots, \lambda_r \in \mathfrak{h}^*$ such that $wt(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_r)$ where $D(\lambda_i) = \{ \lambda_i - \sum_{i=1}^{n} m_i \alpha_i \mid m_i \in \mathbb{Z}_{\geq 0} \}$

Homomorphisms: $\mathfrak{g}$-module homomorphisms.

Fact: Category $\mathcal{O}$ is closed under finite direct sums and tensor products.
(i.e. $V, W \in \mathcal{O} \Rightarrow V \oplus W \in \mathcal{O}$ & $V \otimes W \in \mathcal{O}$)

Most important $\mathfrak{g}$-modules in $\mathcal{O}$ are the highest weight modules:

Defn: A weight module is a highest weight module if

(1) $\exists\ 0 \neq v_\lambda \in V$ ~~with weights~~ such that $\mathcal{U}(\mathfrak{g}) \cdot v_\lambda = V$

(2) $e_i \cdot v_\lambda = 0 \quad \forall\ i \in I$

(3) $h \cdot v_\lambda = \lambda(h) v_\lambda \quad \forall\ h \in \mathfrak{h}$

(i.e. v_λ is a weight vector of weight λ)

We denote such a $\mathfrak{g}$-module by $V(\lambda)$.