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$A = (a_{ij})_{n\times n}$, symmetrizable GCM

$I = \{1, 2, \cdots, n\}$

$D = \mathrm{diag}(s_1, s_2, \cdots, s_n)$

$\mathfrak{g} = \mathfrak{g}(A)$ Kac-Moody Lie alg.

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalg.

$\dim \mathfrak{h} = n + \mathrm{corank}(A)$

$\mathfrak{h} = \mathrm{span}\{h_1, \cdots, h_n, d_1, \cdots, d_t\}$, $t = \mathrm{corank}(A)$

$(\mid) : \mathfrak{h}\times\mathfrak{h} \longrightarrow \mathbb{C} = F$

by $(h_i \mid h) = s_i^{-1}\alpha_i(h) \quad \forall\, h \in \mathfrak{h}$

$(d_i \mid d_j) = 0 \quad , \quad 1 \le i, j \le t.$

$(\mid)$ nondeg. symm. bilinear form on $\mathfrak{h}$.

Define vect. sp. isom.

$$\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$$

by $\nu(h)(h') := (h \mid h') \quad \forall\, h, h' \in \mathfrak{h}$.

Observe

$$\nu(h_i)(h') = (h_i \mid h') = s_i^{-1}\alpha_i(h') \;\forall\, h' \in \mathfrak{h}$$

$$\Longrightarrow \quad \nu(h_i) = s_i^{-1}\alpha_i \quad , \quad i = 1, 2, \cdots, n.$$

$$\Rightarrow \quad \nu^{-1}(\alpha_i) = s_i h_i \,, \qquad 1 \le i \le n .$$

$$\begin{aligned}
\Rightarrow (\alpha_i \mid \alpha_j) &= (s_i h_i \mid s_j h_j) \\
&= \cancel{s_i} s_j \cancel{s_i^{-1}} \alpha_i(h_j) = s_j a_{ji} \\
&= s_i a_{ij} = (\alpha_j \mid \alpha_i)
\end{aligned}$$

Define the symmetric bilinear form

$$(\ \mid\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathbb{C}$$

by $\quad (\alpha \mid \beta) = (\nu^{-1}\alpha \mid \nu^{-1}\beta) \quad \forall\, \alpha, \beta \in \mathfrak{h}^*$

In particular, $(\alpha_i \mid \alpha_j) = (s_i h_i \mid s_j h_j)$

$$= s_j a_{ji} .$$

Then $(\ \mid\ )$ is nondegenerate.

Recall the Weyl group for $\mathfrak{g}$:

$$W = \langle r_1, r_2, \cdots, r_n \rangle$$

$$r_i(\mu) = \mu - \mu(h_i)\alpha_i \quad \forall\ \mu \in \mathfrak{h}^*$$

We want to show that $(\ \mid\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathbb{C}$ is W-invariant, i.e.

$$(w\mu \mid w\mu') = (\mu \mid \mu') \quad \forall\, \mu, \mu' \in \mathfrak{h}^*$$

Since $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ (reduced form)

$$(r_k\alpha_i \mid r_k\alpha_j) \overset{?}{=} (\alpha_i \mid \alpha_j)$$

$$\text{LHS} = (\alpha_i - \alpha_i(h_k)\alpha_k \mid \alpha_j - \alpha_j(h_k)\alpha_k)$$

$$= (\alpha_i \mid \alpha_j) - \alpha_j(h_k)(\alpha_i \mid \alpha_k) - \alpha_i(h_k)(\alpha_k \mid \alpha_j)$$

$$+ \alpha_i(h_k)\alpha_j(h_k)(\alpha_k \mid \alpha_k)$$

$$= (\alpha_i \mid \alpha_j) - s_k a_{ki} a_{kj} - \underbrace{s_j a_{jk}}_{s_k a_{kj}} a_{ki}$$

$$+ a_{ki} a_{kj} s_k a_{kk}$$

$$= (\alpha_i \mid \alpha_j) - s_k a_{ki} a_{kj} - s_k a_{kj} a_{ki}$$

$$+ 2 s_k a_{ki} a_{kj}$$

$$= (\alpha_i \mid \alpha_j)$$

<u>Exer.</u> Using $(\mu \mid \mu') = (\bar{\nu}^{-1}\mu \mid \bar{\nu}^{-1}\mu')$ show that $(r_k\mu \mid r_k\mu') = (\mu \mid \mu')$.

Now we extend the nondeg. symm. bil form

$$(\mid) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$$

to a nondeg. symmetric bil. form

$$(\mid) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$$

such that

(1) $(\mathfrak{g}_\alpha \mid \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$

(2) $(\mid) : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \longrightarrow \mathbb{C}$ is nondeg. and

$$[x, y] = (x \mid y)\, \nu^{-1}(\alpha) \quad \forall\, x \in \mathfrak{g}_\alpha,\, y \in \mathfrak{g}_{-\alpha}$$

(3) $([x, y] \mid z) = (x \mid [y, z]) \quad \forall\, x, y, z \in \mathfrak{g}$

(i.e. $(\mid)$ is associative (= invariant).

Recall $\check{P} = \{h_1, \dots, h_n, d_1, \dots, d_t\}$

and $\mathfrak{h} = \operatorname{span}_{\mathbb{C}} \{\check{P}\}$

Choose $\rho \in \mathfrak{h}^*$ such that

$$\rho(h_i) = 1, \quad i = 1, 2, \dots, n.$$

Ex:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{sl}(3, \mathbb{C})$$

$$\mathfrak{h} = \text{span}\{\underbrace{E_{11} - E_{22}}_{\substack{\shortparallel \\ h_1}}, \underbrace{E_{22} - E_{33}}_{h_2}\}$$

$$\mathfrak{h}^* = \text{span}_{\mathbb{C}}\{\alpha_1, \alpha_2\}$$

$$\alpha_1(h_1) = 2 = \alpha_2(h_2)$$

$$\alpha_1(h_2) = -1 = \alpha_2(h_1)$$

$$\rho \overset{?}{\doteq} \underbrace{\frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha}_{\substack{\shortparallel \\ \frac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)) = \alpha_1 + \alpha_2 \in P}} \in \mathfrak{h}^*$$

$$\rho(h_1) = (\alpha_1 + \alpha_2)(h_1) = 2 - 1 = 1$$

$$\rho(h_2) = (\alpha_1 + \alpha_2)(h_2) = 2 - 1 = 1$$

Ex:

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}, \quad \mathfrak{g} = \mathfrak{g}(A) = \mathfrak{sl}(n, \mathbb{C})$$

$$\Delta_+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \mid 1 \le i < j \le n\}$$

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha .$$ Show that $\rho(h_i) = 1$ for $i = 1, 2, \cdots, n-1$.

Casimir Operator:

Choose ordered bases $\{u_i\}$ and $\{u^i\}$ for $\mathfrak{h}$ such that

$$(u_i \mid u^j) = \delta_{ij}$$

For each $\alpha \in \Delta_+$, choose ordered basis $\{e_\alpha^{(i)}\}$ for $\mathfrak{g}_\alpha$ and $\{f_\alpha^{(i)}\}$ for $\mathfrak{g}_{-\alpha}$ such that

$$\left(e_\alpha^{(i)} \mid f_\alpha^{(j)}\right) = \delta_{ij}$$

Define the Casimir operator to be the formal sum:

$$\Omega = 2\nu^{-1}(\rho) + \sum_{i=1}^{n} u_i u^i + 2\sum_{\alpha\in\Delta_+}\sum_{j=1}^{\dim \mathfrak{g}_\alpha} f_\alpha^{(j)} e_\alpha^{(j)} .$$

Ex: $A = (2)$, $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$

$\mathfrak{h} = \operatorname{span}\{h\}$, $\Delta = \{\alpha, -\alpha\}$

$\mathfrak{g}_\alpha = \operatorname{span}\{e\}$, $\mathfrak{g}_{-\alpha} = \operatorname{span}\{f\}$

$(\mid) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$

$(h \mid h) = 2 = \operatorname{trace}(h^2) \Rightarrow u_1 = h, \; u^1 = \frac{1}{2}h$

$$(e \mid f) = \text{trace}(ef) = 1$$

$$\rho = \tfrac{1}{2}\alpha \ , \quad \nu^{-1}(\rho) = \nu^{-1}(\tfrac{1}{2}\alpha) = \tfrac{1}{2}h$$

$$\Longrightarrow \Omega = 2\nu^{-1}(\rho) + u_1 u^1 + 2fe$$

$$= 2\left(\tfrac{1}{2}h\right) + \tfrac{1}{2}h^2 + 2fe$$

$$= \tfrac{1}{2}h^2 + \underbrace{(h + fe)}_{\overset{\shortparallel}{ef}} + fe$$

$$= \tfrac{1}{2}h^2 + ef + fe. \qquad \checkmark$$