$A = (a_{ij})_{i,j \in I}$ symmetrizable GCM

$D = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ nonsingular such that $DA = (s_i a_{ij})_{i,j \in I}$ symmetric.

$(\mathfrak{h}, \Pi, \check{\Pi}, P, \check{P})$ Cartan datum.

$\check{P} = \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d_1 \oplus \cdots \oplus \mathbb{Z}d_s$

$s = \mathrm{corank}(A)$.

$\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$ Cartan subalg.

$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h) \in \mathbb{Z} \ \forall h \in \check{P}\}$

$\Pi = \{\alpha_i \in \mathfrak{h}^* \mid i \in I\}$ simple roots

$\alpha_j(h_i) = a_{ij}$

$\check{\Pi} = \{h_i \in \mathfrak{h} \mid i \in I\}$.

$\mathfrak{g} = \mathfrak{g}(A)$ Kac-Moody Lie algebra.

$\mathcal{U}(\mathfrak{g})$ = universal enveloping alg. of $\mathfrak{g}$
$= \langle e_i, f_i \mid i \in I \rangle \cup \check{P}$.

$\mathcal{U}(\mathfrak{g})$ is a Hopf algebra with

multiplication : $\mu : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$

$x \otimes y \longmapsto x \cdot y$

assoc. prod.

unit ; $\eta : \mathbb{C} \longrightarrow \mathcal{U}(\mathfrak{g})$

$1 \longmapsto 1$

comultiplication : $\Delta : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$

$x \longmapsto x \otimes 1 + 1 \otimes x$

$\forall x \in \mathfrak{g}$

counit : $\varepsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}$

$x \longmapsto 0$

$\mu, \eta, \Delta, \varepsilon$ are algebra homomorphisms.

antipode : $S : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$

$x \longmapsto -x \quad \forall x \in \mathfrak{g}$

and S is an antihomomorphism.

Set $F = \mathbb{C}(q)$

The quantum group $U_q(\mathfrak{g})$ with Cartan datum $(A, \mathfrak{h}, \Pi, \check{\Pi}, P, \check{P})$ is the associative algebra with unit over F generated by $\{e_i, f_i \mid i \in I\} \cup \check{P}$ satisfying the following relations:

(1) $q^0 = 1, \quad q^h q^{h'} = q^{h'} q^h = q^{h+h'} \quad \forall\, h, h' \in \check{P}$

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad i \in I, \ h \in \check{P}$

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \quad i \in I, \ h \in \check{P}$

(4) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad \forall\, i, j \in I$

(Here $q_i = q^{s_i}$, $t_i = q_i^{h_i} = q^{s_i h_i}$)

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Remark: Define

$$(\mathrm{ad}_q\, x)(y) = xy - q^{(\alpha|\beta)}\, yx$$

$\forall\ x \in (\mathcal{U}(\mathfrak{g}))_\alpha\ ,\ y \in (\mathcal{U}(\mathfrak{g}))_\beta\ ,\ \alpha, \beta \in Q$

where $(\alpha_i|\alpha_j) = \alpha_j(h_i) = s_i a_{ij}$

Then

$$(\mathrm{ad}_q\, e_i)^N (e_j) = \sum_{k=0}^{N} (-1)^k q_i^{k(N+a_{ij}-1)} \begin{bmatrix} N \\ k \end{bmatrix}_{q_i} e_i^{N-k}\, e_j\, e_i^k$$

$$\Rightarrow (\mathrm{ad}_q\, e_i)^{1-a_{ij}} (e_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k}\, e_j\, e_i^k$$

Hence relations (5) & (6) can be written as

$$(\mathrm{ad}_q\, e_i)^{1-a_{ij}} e_j = 0 \quad , \quad i \neq j$$

$$(\mathrm{ad}_q\, f_i)^{1-a_{ij}} f_j = 0 \quad , \quad i \neq j$$

respectively.

The quantum group $U_q(\mathfrak{g})$ is a Hopf algebra with

Multiplication: $\mu : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$

$$x \otimes y \longmapsto x \cdot y$$

(assoc. product)

Unit : $\eta : F \longrightarrow U_q(\mathfrak{g})$

$$1 \longmapsto 1$$

Comultiplication: $\Delta : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$

where $\Delta(q^h) = q^h \otimes q^h \quad \forall\, h \in \check{P}$

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i \;,\; i \in I$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i \;,\; i \in I$$

Counit : $\varepsilon : U_q(\mathfrak{g}) \longrightarrow F$

where $\varepsilon(q^h) = 1 \;,\; \varepsilon(e_i) = 0 = \varepsilon(f_i)$

$\forall\, h \in \check{P},\; i \in I$

Antipode : $S : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$

where $S(q^h) = q^{-h} \,,\, S(e_i) = -e_i t_i \,,\, S(f_i) = -t_i^{-1} f_i$

$A = (a_{ij})_{n\times n}$ symmetrizable GCM

$D = \mathrm{diag}(s_1, s_2, \cdots, s_n)$, DA symmetric

Note $s_i a_{ij} = s_j a_{ji}$ $\forall\, i, j \in I$.

$(\mathfrak{h}, \Pi, \check{\Pi}, P, \check{P})$ Cartan datum

$\mathfrak{g} = \mathfrak{g}(A)$ Kac-Moody Lie algebra

<u>Goal</u>: Define a nondegenerate, invariant, symmetric bilinear form on $\mathfrak{g}$.

~~$\check{P} = \ldots$~~

$\check{P} = \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d_1 \oplus \cdots \oplus \mathbb{Z}d_s$

$s = \mathrm{corank}(A)$

$\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$, $\dim(\mathfrak{h}) = n + \mathrm{corank}(A)$

$\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{h_1, \cdots, h_n, d_1, \cdots, d_s\}$

Define $(\ |\ ) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$

by $(h_i | h) = s_i^{-1} \alpha_i(h)$ $\forall\, h \in \mathfrak{h}, i \in I$

$(d_i | d_j) = 0$ $\forall\, i, j = 1, 2, \cdots, s$.

$\Rightarrow$ (|) is a nondeg. symmetric bilinear form.

Note $(h_i | h_j) = s_i^{-1} \alpha_i(h_j) = s_i^{-1} a_{ji}$

$$= s_i^{-1} s_i s_j^{-1} a_{ij} = s_j^{-1} a_{ij}$$

$$= s_j^{-1} \alpha_j(h_i) = (h_j | h_i)$$