$A = (a_{ij})_{i,j \in I}$ symmetrizable GCM
$(I = \{1, 2, \cdots, n\})$

- $a_{ii} = 2$
- $a_{ij} \le 0$, $i \ne j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
- $\exists$ nonsingular $D = \mathrm{diag}(s_1, s_2, \cdots, s_n)$ such that DA is symmetric.

$k = \mathrm{corank}(A)$

Fix free abelian group (i.e. integral lattice)

$$\check{P} = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d_1 \oplus \cdots \oplus \mathbb{Z}d_k$$

called "dual weight lattice"

Set (1) $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$ called "Cartan subalg".

$= \mathrm{span}_{\mathbb{C}} \{h_1, \cdots, h_n, d_1, \cdots, d_k\}$

(2) $\check{\Pi} = \{h_i \mid i \in I\} \subseteq \mathfrak{h}$ "simple coroots".

(3) A lin. indep. set $\Pi = \{\alpha_i \mid i \in I\} \subseteq \mathfrak{h}^*$, called "simple roots" satisfying

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_l) = 0 \text{ or } 1.$$

(4) $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h) \in \mathbb{Z}\}$ called "weight lattice"

$(\Pi, \check{\Pi}, P, \check{P})$ called "Cartan datum" associated with $A = (a_{ij})_{i,j \in I}$.

Define
$$Q := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \cdots \oplus \mathbb{Z}\alpha_n$$
called the "root lattice"

$$Q_+ := \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \cdots + \mathbb{Z}_{\geq 0}\alpha_n$$
called the "positive root lattice".

$$Q_- := -Q_+$$
Note that $Q \neq Q_+ \cup Q_-$

For $\lambda, \mu \in \mathfrak{h}^*$ define

$$\lambda \geq \mu \quad \text{if} \quad \lambda - \mu \in Q_+$$
'$\geq$' is a "partial order" on $\mathfrak{h}^*$

Define $\Lambda_j \in \mathfrak{h}^*$, $j \in I$ by

$$\Lambda_j(h_i) = \delta_{ij}, \quad \Lambda_j(d_\ell) = 0$$
$\{\Lambda_1, \Lambda_2, \cdots, \Lambda_n\}$ is the set of "fundamental weights. Note

$$\mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \subseteq P.$$

For $i \in I$, we define "simple reflection"
$r_i : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$
Easy to see that $r_i^2 = id$ and r_i is an automorphism.

Define the "Weyl group" to be the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{r_1, r_2, \cdots, r_n\}$.
For $w \in W$, we can write

$$W = r_{i_1} r_{i_2} \cdots r_{i_t} \quad , \; i_j \in I$$

t is called the "length of w" denoted by $l(w)$ if t is the smallest positive number among all such expressions of w.

<u>Kac-Moody alg. $\mathfrak{g} = \mathfrak{g}(A)$:</u>

$\mathfrak{g}$ is the Lie algebra generated by
$\{e_i, f_i \mid i \in I\} \cup \check{P}$ subject to the relations:

(1) $[h, h'] = 0 \;\; \forall \; h, h' \in \check{P}$

(2) $[e_i, f_j] = \delta_{ij} h_i \;\; , \;\; i, j \in I$

(3) $[h, e_j] = \alpha_j(h)\, e_j \;\; \forall \; h \in \check{P}, \; j \in I$

(4) $[h, f_j] = -\alpha_j(h) f_j$, $h \in \check{P}$, $j \in I$

(5) $(ad_{e_i})^{1-a_{ij}} e_j = 0$, $i \neq j$, $i, j \in I$

(6) $(ad_{f_i})^{1-a_{ij}} f_j = 0$, $i \neq j$, $i, j \in I$

The subalg. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$ is called the derived subalg. of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is said to be <u>perfect</u> if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Define

$$\mathfrak{g}_+ = \text{subalg. gen. by } \{e_i \mid i \in I\}$$

$$\mathfrak{g}_- = \text{subalg. gen. by } \{f_i \mid i \in I\}$$

<u>Thm:</u> $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$. (triangular decomp.)

For $\alpha \in Q$ define

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) x \ \forall\, h \in \check{P}\}$$

A nonzero $\alpha \in Q$ is a <u>root</u> if the α-root space $\mathfrak{g}_\alpha \neq 0$.

Denote Δ = set of roots of $\mathfrak{g}$

$\alpha \in \Delta$, $\alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n$

where <u>all</u> $m_i \in \mathbb{Z}_{\geq 0}$ or <u>all</u> $m_i \in \mathbb{Z}_{\leq 0}$

Then $\Delta = \Delta_+ \cup \Delta_-$ (disjoint)

Δ_+ = set of positive roots

$\Delta_- = -\Delta_+$ = set of negative roots.

<u>Prop</u>: $\mathfrak{g}_{\pm} = \bigoplus_{\alpha \in \Delta_{\pm}} \mathfrak{g}_\alpha$ and $\dim \mathfrak{g}_\alpha < \infty$.

We have an involution $\eta : \mathfrak{g} \to \mathfrak{g}$ defined by $\eta(e_i) = -f_i$, $\eta(f_i) = -e_i$ $\forall i \in I$ and $\eta(h) = -h$ $\forall h \in \mathfrak{h}$.

called "Chevalley involution".

Hence

$$\eta(\mathfrak{g}_\pm) = \mathfrak{g}_\mp$$

<u>Thm</u>: The center of $\mathfrak{g}$ is given by

$$Z(\mathfrak{g}) = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \ \forall i \in I\}$$

$\Rightarrow \dim Z(\mathfrak{g}) = \operatorname{corank}(A)$.

$\mathcal{U}(\mathfrak{g})$ = universal enveloping alg. of $\mathfrak{g}$.

Recall $x, y \in \mathcal{U}(\mathfrak{g})$

$$(\mathrm{ad}_x)^N(y) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} x^{N-k} y x^k .$$

<u>Prop</u>: $\mathcal{U}(\mathfrak{g})$ is the associative alg. with unity, over $\mathbb{C}$ gen. by $\{e_i, f_i \mid i \in \mathfrak{h}\} \cup \{\check{P}\}$ subject to the relations:

(1) $hh' = h'h \quad \forall\, h, h' \in \check{P}$

(2) $e_i f_j - f_j e_i = \delta_{ij} h_i, \quad \forall\, i, j \in I$

(3) $he_i - e_i h = \alpha_i(h) e_i, \quad \forall\, i \in I,\ h \in \check{P}$

(4) $hf_i - f_i h = -\alpha_i(h) f_i, \quad \forall\, i \in I,\ h \in \check{P}$

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \forall\, i \neq j$

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad \forall\, i \neq j.$

Let

$U^+ =$ subalg of $\mathcal{U}(\mathfrak{g})$ gen. by $e_i, i \in I$

$U^- =$ subalg of $\mathcal{U}(\mathfrak{g})$ gen. by $f_i, i \in I$

$U^0 =$ subalg of $\mathcal{U}(\mathfrak{g})$ gen. by $\check{P}$

<u>Prop</u>: $\mathcal{U}(\mathfrak{g}) = U^- \otimes U^0 \otimes U^+$.

For $\beta \in Q$ define

$$U_\beta = \{ u \in \mathcal{U}(\mathfrak{g}) \mid hu - uh = \beta(h)u \ \forall h \in \check{P} \}$$

$$U^\pm_\beta = U^\pm \cap U_\beta .$$

Then we have:

<u>Prop</u>: (1) $\mathcal{U}(\mathfrak{g}) = \bigoplus_{\beta \in Q} U_\beta$

(2) $U^\pm = \bigoplus_{\beta \in Q_\pm} U^\pm_\beta$