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Define maps

$\varphi, \varepsilon : B(\ell) \longrightarrow \mathbb{Z}$ by

$$\varepsilon(u_i^{(\ell)}) = \#\text{ of arrows coming into } u_i^{(\ell)} = i$$

$$\varphi(u_i^{(\ell)}) = \#\text{ of arrows going out from } u_i^{(\ell)} = \ell - i$$

Observe
$$\begin{aligned} wt(u_i^{(\ell)}) &= \ell - 2i \\ &= (\ell - i) - i \\ &= \varphi(u_i^{(\ell)}) - \varepsilon(u_i^{(\ell)}) \end{aligned}$$

The action of the Kashiwara operators $\tilde{e}$ and $\tilde{f}$ on $V(\ell) \otimes V(m)$:

Example: $V(2) \otimes V(1)$

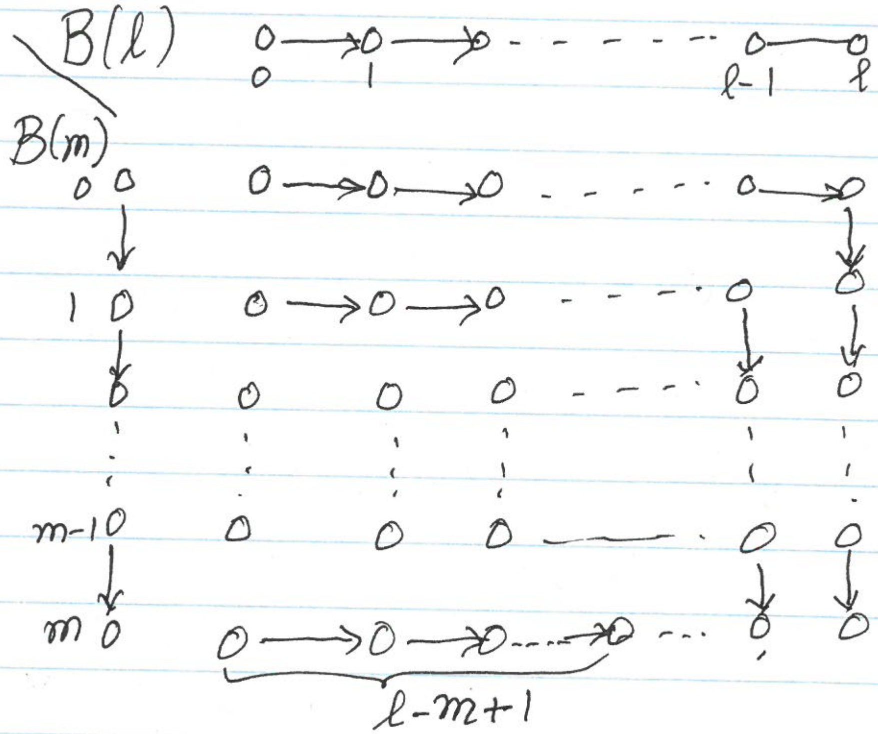
$$B(2):\ u_0^{(2)} \longrightarrow u_1^{(2)} \longrightarrow u_2^{(2)}$$

$$\begin{tikzcd} B(1) & & & \\ v_0^{(1)} \arrow[d] & u_0^{(2)} \otimes v_0^{(1)} \arrow[r, "\tilde{f}"] & u_1^{(2)} \otimes v_0^{(1)} \arrow[r, "\tilde{f}"] & u_2^{(2)} \otimes v_0^{(1)} \arrow[d, "\tilde{f}"] \\ v_1^{(1)} & u_0^{(2)} \otimes v_1^{(1)} \arrow[r, "\tilde{f}"] & u_1^{(2)} \otimes v_1^{(1)} & u_2^{(2)} \otimes v_1^{(1)} \end{tikzcd}$$

$$\tilde{f}(u_i^{(\ell)} \otimes v_j^{(m)}) = \begin{cases} \tilde{f} u_i^{(\ell)} \otimes v_j^{(m)} & \text{if } \varphi(u_i^{(\ell)}) > \varepsilon(v_j^{(m)}) \\ u_i^{(\ell)} \otimes \tilde{f} v_j^{(m)} & \text{if } \varphi(u_i^{(\ell)}) \leq \varepsilon(v_j^{(m)}) \end{cases}$$

$$\tilde{e}\left(u_i^{(\ell)} \otimes v_j^{(m)}\right) = \begin{cases} \tilde{e}\left(u_i^{(\ell)}\right) \otimes v_j^{(m)} & \text{if } \varphi\left(u_i^{(\ell)}\right) \geq \varepsilon\left(v_j^{(m)}\right) \\ u_i^{(\ell)} \otimes \tilde{e}\, v_j^{(m)} & \text{if } \varphi\left(u_i^{(\ell)}\right) < \varepsilon\left(v_j^{(m)}\right) \end{cases}$$

$B(\ell) \otimes B(m)$ for $V(\ell) \otimes V(m)$, $\ell \geq m$



$$B(\ell) \otimes B(m) = B(\ell+m) \oplus B(\ell+m-2) \oplus \cdots \oplus B(\ell-m)$$

$$\Rightarrow V(\ell) \otimes V(m) \cong V(\ell+m) \oplus V(\ell+m-2) \oplus \cdots \oplus V(\ell-m)$$

$A = (a_{ij})_{i,j \in I}$ symmetrizable GCM:

- $a_{ii} = 2$
- $a_{ij} \leq 0$, $i \neq j$
- $a_{ij} = 0 \iff a_{ji} = 0$
- $\exists$ diag. matrix $D = \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \end{pmatrix}$, $s_i \in \mathbb{Z}_{>0}$ such that DA symmetric.

The Kac-Moody algebra

~~$\mathfrak{g} = \mathfrak{g}(A) = \langle \{e_i, f_i\}_{i \in I} \rangle$~~

$$\mathfrak{g} = \mathfrak{g}(A) = \langle \mathfrak{h}, e_i, f_i \mid i \in I \rangle, h_i = [e_i, f_i]$$

where $\mathfrak{h}$ is abelian of dimension $|I| + \text{corank}(A)$ satisfying:

(1) $[h, h'] = 0 \quad \forall h, h' \in \mathfrak{h}$

(2) $[h_i, e_j] = a_{ij} e_j$, $i, j \in I$

(3) $[h_i, f_j] = -a_{ij} f_j$, $i, j \in I$

(4) $[e_i, f_j] = \delta_{ij} h_i$, $i, j \in I$

(5) $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 \,, \quad i \neq j \,, \; i,j \in I$

(6) $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \,, \quad i \neq j \,, \; i,j \in I .$

$\mathcal{U}(g) =$ universal enveloping alg. of g.

For $x, y \in \mathcal{U}(g)$

$$\mathrm{ad}_x(y) = [x, y] = xy - yx$$

$$\begin{aligned}
\mathrm{ad}_x^2(y) &= \mathrm{ad}_x(xy - yx) \\
&= \cancel{\mathrm{ad}_x(x)}\, y + x\, \mathrm{ad}_x(y) \\
&\quad - \mathrm{ad}_x(y)\, x - y\, \cancel{\mathrm{ad}_x(x)} \\
&= x(xy - yx) - (xy - yx)x \\
&= x^2 y - 2xyx + yx^2 \\
&= x^2 y x^0 - 2xyx + x^0 y x^2 \\
&= \sum_{k=0}^{2} (-1)^k \binom{2}{k} x^{2-k} y x^k
\end{aligned}$$

In general, $\mathrm{ad}_x^N(y) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} x^{N-k} y x^k .$