Exer. Show that the highest weight vector

$$w_0^{(\ell+m-2k)}, \quad 0 \leq k \leq m$$

of the irreducible component $V(\ell+m-2k)$ in $V(\ell) \otimes V(m)$ is given by:

$$w_0^{(\ell+m-2k)} = \sum_{i=0}^{k} (-1)^i \frac{[\ell-i]!\,[m-k+i]!}{[\ell]!\,[m-k]!} \cdot q^{i(m-2k+i+1)} \left(u_i^{(\ell)} \otimes v_{k-i}^{(m)}\right)$$

where $V(\ell) = \text{span}\{u_0^{(\ell)}, u_1^{(\ell)}, \cdots, u_\ell^{(\ell)}\}$

and $V(m) = \text{span}\{v_0^{(m)}, v_1^{(m)}, \cdots, v_m^{(m)}\}$.

Pf: (Hint) Show that

$$e \cdot w_0^{(\ell+m-2k)} = 0 \quad \text{and}$$

$$t \cdot w_0^{(\ell+m-2k)} = q^{(\ell+m-2k)}\, w_0^{(\ell+m-2k)}.$$

Example: By the q-Clebsh-Gordon formula

$$V(2) \otimes V(1) \cong V(3) \oplus V(1)$$

$$V(2) = \operatorname{span}\{u_0^{(2)}, u_1^{(2)}, u_2^{(2)}\}$$

$$V(1) = \operatorname{span}\{v_0^{(1)}, v_1^{(1)}\}$$

$$w_0^{(3)} = \frac{[2]!\,[1]!}{[2]!\,[1]!} q^0 u_0^{(2)} \otimes v_0^{(1)} = u_0^{(2)} \otimes v_0^{(1)}$$

$$w_0^{(1)} = \frac{[2]!\,[0]!}{[2]!\,[0]!} q^0 u_0^{(2)} \otimes v_1^{(1)}$$

$$+ (-1) \frac{[1]!\,[1]}{[2]!\,[0]!} q^1 u_1^{(2)} \otimes v_0^{(1)}$$

$$= u_0^{(2)} \otimes v_1^{(1)} - q[2]^{-1} u_1^{(2)} \otimes v_0^{(1)}$$

$$= u_0^{(2)} \otimes v_1^{(1)} - \frac{q}{q+q^{-1}} u_1^{(2)} \otimes v_0^{(1)}$$

Hence

$$V(3) = \operatorname{span}\{w_0^{(3)}, f^{(1)} w_0^{(3)}, f^{(2)} w_0^{(3)}, f^{(3)} w_0^{(3)}\}$$

and

$$V(1) = \operatorname{span}\{w_0^{(1)}, f\, w_0^{(1)}\}$$

Recall $f^{(k)} = \dfrac{f^k}{[k]!}$

Recall $K = \mathbb{C}(q)$.

Define the subring of K:

$$A = \left\{ f(q) = \frac{g(q)}{h(q)} \in K \;\middle|\; \begin{array}{l} g(q), h(q) \in \mathbb{C}[q] \\ \& \; h(0) \neq 0 \end{array} \right\}$$

Then A is an integral domain and its fraction field is K.

Consider the evaluation map:

$$A/qA \longrightarrow \mathbb{C}$$

$$f(q) + qA \longmapsto f(0)$$

is an isomorphism.

For the irred. $U_q(sl(2))$-module

$$V(\ell) = \operatorname{span}\{ u_i^{(\ell)} \mid 0 \le i \le \ell \}$$

define the free A-lattice

$$L(\ell) = \bigoplus_{i=0}^{\ell} A u_i^{(\ell)}$$

Then $K \otimes_A L(\ell) \cong V(\ell)$.

Define the set

$$B(\ell)=\left\{ u_i^{(\ell)} = f^{(i)} u_0^{(\ell)} = \tilde{f}^{\,i} u_0^{(\ell)} \,\middle|\, 0 \le i \le \ell \right\} \subset L(\ell)/qL(\ell)$$

is called the <u>crystal</u> for $V(\ell)$.

The lattice $L(\ell)$ is called the crystal lattice.

Recall $V(2) \otimes V(1) \cong V(3) \oplus V(1)$

Denote the crystal

$$B(2)=\{u_0^{(2)}, u_1^{(2)}, u_2^{(2)}\} \text{ of } V(2)$$

and $B(1)=\{v_0^{(1)}, v_1^{(1)}\}$ of $V(1)$.

What is the crystal for $V(3)$ & $V(1)$?

$$V(3) = \operatorname{span}_K\{w_0^{(3)}, f w_0^{(3)}, f^{(2)} w_0^{(3)}, f^{(3)} w_0^{(3)}\}$$

exer.

$$= \operatorname{span}_K\{u_0^{(2)} \otimes v_0^{(1)},\ u_1^{(2)} \otimes v_0^{(1)} + q^2 u_0^{(2)} \otimes v_1^{(1)},$$

$$u_2^{(2)} \otimes v_0^{(1)} + q u_1^{(2)} \otimes v_1^{(1)},\ u_2^{(2)} \otimes v_1^{(1)}\}$$

$$V(1) \overset{\text{expr.}}{=} \operatorname{span}_K \left\{ u_0^{(2)} \otimes v_1^{(1)} - \frac{q^2}{q^2+1} u_1^{(2)} \otimes v_0^{(1)}, \right.$$
$$\left. u_1^{(2)} \otimes v_1^{(1)} - q\, u_2^{(2)} \otimes v_0^{(1)} - \frac{q^2}{q^2+1} u_1^{(2)} \otimes v_1^{(1)} \right\}$$

Let L denote the crystal lattice of $V(2) \otimes V(1)$. Then

$$L = \bigoplus_{\substack{0 \le i \le 2 \\ 0 \le j \le 1}} A\, u_i^{(2)} \otimes v_j^{(1)}$$

and the crystal for $V(2) \otimes V(1)$ is

$$B = \left\{ u_i^{(2)} \otimes v_j^{(1)} \mid 0 \le i \le 2,\ 0 \le j \le 1 \right\}$$

$$B(3) \subset L(3) / qL(3)$$

$$\Rightarrow B(3) = \left\{ u_0^{(2)} \otimes v_0^{(1)},\ u_1^{(2)} \otimes v_0^{(1)},\ u_2^{(2)} \otimes v_0^{(1)},\ u_2^{(2)} \otimes v_1^{(1)} \right\}$$

$$\Rightarrow L(3) = A(u_0^{(2)} \otimes v_0^{(1)}) \oplus A(u_1^{(2)} \otimes v_0^{(1)})$$
$$\oplus A(u_2^{(2)} \otimes v_0^{(1)}) \oplus A(u_2^{(2)} \otimes v_1^{(1)})$$

$B'(1)$ be the crystal for $V(1)$ (in $V(3) \oplus V(1)$)

Then $B'(1) = \left\{ u_0^{(2)} \otimes v_1^{(1)},\ u_1^{(2)} \otimes v_1^{(1)} \right\}$

and $L'(1) = A(u_0^{(2)} \otimes v_1^{(1)}) \oplus A(u_1^{(2)} \otimes v_1^{(1)})$.

Hence we observe that

$$L = L(3) \oplus L'(1) \quad \text{as } A\text{-modules} \quad \text{and}$$
$$B = B(3) \cup B'(1).$$

For the irred. $U_q(sl(2))$-module $V(\ell) = \text{span}\{u_i^{(\ell)} \mid 0 \le i \le \ell\}$ with crystal lattice $L(\ell) = \oplus A u_i^{(\ell)}$ and crystal $B(\ell) = \{u_i^{(\ell)} \mid 0 \le i \le \ell\} \subset L(\ell)/qL(\ell)$ we define homomorphisms

$$\tilde{e}, \tilde{f} : V(\ell) \longrightarrow V(\ell)$$

by $\quad \tilde{f} u_i^{(\ell)} = u_{i+1}^{(\ell)}$ and $\tilde{e} u_i^{(\ell)} = u_{i-1}^{(\ell)}$.

$\tilde{e}, \tilde{f}$ are called Kashiwara operators.
Note that

$$\tilde{f}(L(\ell)) \subseteq L(\ell)$$
$$\tilde{e}(L(\ell)) \subseteq L(\ell)$$

and

$$\tilde{f}(B(\ell)) \subseteq B(\ell) \cup \{0\}$$
$$\tilde{e}(B(\ell)) \subseteq B(\ell) \cup \{0\}$$

The graph: $u_0^{(\ell)} \xrightarrow{\tilde{f}} u_1^{(\ell)} \xrightarrow{\tilde{f}} \cdots \xrightarrow{\tilde{f}} u_\ell^{(\ell)}$
is called the crystal graph of $V(\ell)$.