Finite dim'l $U_q(sl(2))$-modules:

For any $l \in \mathbb{Z}_{\geq 0}$ define a $(l+1)$-dim'l vector space $V(l)$ with basis:

$$\{u_0^{(l)}, u_1^{(l)}, ----, u_l^{(l)}\}$$

Define $U_q(sl(2))$-action on $V(l)$ by:

$$t^{\pm 1} \cdot u_k^{(l)} = q^{\pm(l-2k)} u_k^{(l)}$$

$$e \cdot u_k^{(l)} = [l-k+1] u_{k-1}^{(l)}$$

$$f \cdot u_k^{(l)} = [k+1] u_{k+1}^{(l)}$$

Thm: $V(l)$ is an irred. $U_q(sl(2))$-module.

Pf: Need to check:

✓ (1) $(tet^{-1}) \cdot u_k^{(l)} = (q^2 e) \cdot u_k^{(l)}$

(2) $(tft^{-1}) \cdot u_k^{(l)} = (q^{-2} f) \cdot u_k^{(l)}$ (exer.)

✓ (3) $(ef - fe) \cdot u_k^{(l)} = \left(\frac{t - t^{-1}}{q - q^{-1}}\right) \cdot u_k^{(l)}$

(1) $(te) \cdot t^{-1} \cdot u_k^{(l)} = (te) \cdot q^{-(l-2k)} u_k^{(l)}$

$$= t \cdot e \cdot q^{-(l-2k)} u_k^{(l)} = q^{-(l-2k)} t \cdot [l-k+1] u_{k-1}^{(l)}$$

$$= q^{-(l-2k)} [l-k+1] \, q^{(l-2k+2)} u_{k-1}^{(l)}$$

$$= q^{2} [l-k+1] u_{k-1}^{(l)} = q^{2} e \cdot u_{k} .$$

(3) $(ef - fe) \cdot u_k^{(l)} = e \cdot f \cdot u_k^{(l)} - f \cdot e \cdot u_k^{(l)}$

$$= e \cdot [k+1] u_{k+1}^{(l)} - f \cdot [l-k+1] u_{k-1}^{(l)}$$

$$= [k+1][l-k] u_k^{(l)} - [l-k+1][k] u_k^{(l)}$$

$$= \left([k+1][l-k] - [l-k+1][k] \right) u_k^{(l)}$$

$$= \left(\left(\frac{q^{k+1} - q^{-k-1}}{q - q^{-1}} \right) \left(\frac{q^{l-k} - q^{-(l-k)}}{q - q^{-1}} \right) - \left(\frac{q^{l-k+1} - q^{-(l-k+1)}}{(q - q^{-1})} \right) \left(\frac{q^{k} - q^{-k}}{q - q^{-1}} \right) \right) u_k^{(l)}$$

$$= (q - q^{-1})^{-2} \Big(\cancel{q^{(l+1)}} - q^{-(l-2k-1)} - q^{(l-2k-1)} + \cancel{q^{-(l+1)}}$$
$$- \cancel{q^{(l+1)}} + q^{(l-2k+1)} + \cancel{q^{-(l+1)}} + q^{-(l-2k+1)} - \cancel{q^{-(l+1)}} \Big) u_k^{(l)}$$

$$= (q - q^{-1})^{-2} \left(q^{(l-2k)} (q - q^{-1}) - q^{-(l-2k)} (q - q^{-1}) \right) u_k^{(l)}$$

$$= \left(\frac{q^{(l-2k)} - q^{-(l-2k)}}{q - q^{-1}} \right) u_k^{(l)} = \left(\frac{t - t^{-1}}{q - q^{-1}} \right) \cdot u_k .$$

Suppose $0 \neq W$ is a submodule of $V(l)$.

Then $0 \neq \sum_{i=0}^{k} a_i u_i^{(l)} \in W$

Suppose $a_t \neq 0$ and $a_i = 0 \ \forall\, i > t$.

Then by applying e repeatedly (in fact t times) we get $c\, u_0^{(\ell)} \in W$ where $c \neq 0$

$\Rightarrow \quad u_0^{(\ell)} \in W \Rightarrow$ by applying f, $u_k^{(\ell)} \in W$ for $k = 0, 1, \cdots, \ell$.

$\Rightarrow \quad W = V(\ell)$. //

Define

$$V(\ell)_\mu = \{ u \in V(\ell) \mid t \cdot u = q^\mu u \}$$

μ is a weight if $V(\ell)_\mu \neq 0$.

$\Rightarrow$ The weights of $V(\ell)$ are:

$$\ell, \ \ell-2, \ \ell-4, \ \cdots\cdots, \ -\ell$$

and each weight space is one dim'l.

<u>Thm</u>: (1) Any finite dimensional $U_q(sl(2))$ module is completely reducible.

(2) For $l \in \mathbb{Z}_{\geq 0}$, any irreducible $U_q(sl(2))$ module is isomorphic to $V(l)$ or $V(l) \otimes V_-$

Observe that the weight of $u_k^{(l)} \otimes u_- \in V(l) \otimes V_-$ is $t.(u_k^{(l)} \otimes u_-) = \Delta(t)(u_k^{(l)} \otimes u_-)$

$$= (t \otimes t)(u_k^{(l)} \otimes u_-) = (t.u_k^{(l)}) \otimes (t.u_-)$$

$$= q^{(l-2k)} u_k^{(l)} \otimes (-u_-) = -q^{(l-2k)} u_k^{(l)} \otimes u_-$$

But $-q^{(l-2k)}$ can not be a weight.

<u>Thm</u>: Let V be a finite dim'l $U_q(sl(2))$ module. Then the following conditions are equivalent:

(1) $V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu$

(2) $V = \bigoplus_{i=1}^{k} V(l_i)$

Now suppose V is a finite dim'l $U_q(sl(2))$-module which admits a weight space decomposition. For V we define its character by

$$\mathrm{ch}(V) = \sum_{k \in \mathbb{Z}} (\dim V_k)\, x^k \in \mathbb{Z}[x, x^{-1}]$$

In particular, for $l \in \mathbb{Z}_{\geq 0}$

$$\mathrm{ch}\, V(l) = x^l + x^{l-2} + \cdots + x^{-l+2} + x^{l}$$

$$= \frac{x^{l+1} - x^{-l-1}}{x - x^{-1}}$$

and $\{\mathrm{ch}\, V(l)\}_{l \in \mathbb{Z}_{\geq 0}}$ is $\mathbb{Z}$-linearly independent.

Let $V \cong \bigoplus_{j=1}^{k} V(l_j)$. Then

$$\mathrm{ch}(V) = \sum_{j=1}^{k} \mathrm{ch}\, V(l_j)$$

Hence $\{l_j\}_{1 \leq j \leq k}$ can be uniquely determined by the character of V.

Let $V(l)$ and $V(m)$ be two irred. $U_q(sl(2))$-modules and $l \geq m$.

$$\mathrm{ch}\left(V(\ell)\otimes V(m)\right) = \left(\mathrm{ch}(V(\ell))\,\mathrm{ch}(V(m))\right)$$

$$= \left(\frac{x^{\ell+1} - x^{-(\ell+1)}}{x - x^{-1}}\right)\left(\frac{x^{m+1} - x^{-(m+1)}}{x - x^{-1}}\right)$$

$$= \cancel{(x - x^{-1})^{-2}\left(x^{(\ell+m+2)} - x^{(\ell-m)} - x^{-(\ell-m)} + x^{-(\ell+m+2)}\right)}$$

$$= \left(\frac{x^{\ell+1} - x^{-(\ell+1)}}{x - x^{-1}}\right)\left(x^{m} + x^{m-2} + \cdots + x^{-m+2} + x^{-m}\right)$$

$$= \left(\frac{x^{(\ell+m+1)} - x^{-(\ell+m+1)}}{x - x^{-1}}\right) + \left(\frac{x^{(\ell+m-1)} - x^{-(\ell+m-1)}}{x - x^{-1}}\right)$$

$$+ \cdots + \left(\frac{x^{(\ell-m+1)} - x^{-(\ell-m+1)}}{x - x^{-1}}\right)$$

$$= \mathrm{ch}(V(\ell+m)) + \mathrm{ch}(V(\ell+m-2)) + \cdots + \mathrm{ch}(V(\ell-m))$$

$$= \bigoplus_{k=0}^{m} \mathrm{ch}\left(V(\ell+m-2k)\right) = \mathrm{ch}\left(\bigoplus_{k=0}^{m} V(\ell+m-2k)\right)$$

$$\Rightarrow V(\ell)\otimes V(m) = \bigoplus_{k=0}^{m} V(\ell+m-2k)$$

which is the q-Clebsch-Gordon formula.