$U_q(sl(2))$ - module $\mathbb{C}(q)^2 = \text{span}_{\mathbb{C}(q)}\{v_1, v_2\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\quad V_q$

$e.v_1 = 0, \quad e\cdot v_2 = v_1, \quad f\cdot v_1 = v_2, \quad f v_2 = 0$

$t.v_1 = q v_1, \quad t^{-1} v_1 = q^{-1} v_1, \quad t\cdot v_2 = q^{-1} v_2, \quad t^{-1}\cdot v_2 = q v_2.$

$\Delta : U_q(sl(2)) \longrightarrow U_q(sl(2)) \otimes U_q(sl(2))$

$$\left.\begin{aligned} \Delta(e) &= e\otimes t^{-1} + 1\otimes e \\ \Delta(f) &= f\otimes 1 + t\otimes f \\ \Delta(t^{\pm 1}) &= t^{\pm 1}\otimes t^{\pm 1} \end{aligned}\right\} \begin{array}{l} V_q\otimes V_q \text{ is an } U_q(sl(2)) \\ \text{module.} \end{array}$$

$$\begin{aligned} e\cdot(v_1\otimes v_1) &= (\Delta(e))(v_1\otimes v_1) \\ &= (e\otimes t^{-1} + 1\otimes e)(v_1\otimes v_1) \\ &= e.v_1\otimes t^{-1}\cdot v_1 + v_1\otimes e\cdot v_1 = 0 \end{aligned}$$

$$\begin{aligned} t\cdot(v_1\otimes v_1) &= \Delta(t)(v_1\otimes v_1) = (t\otimes t)(v_1\otimes v_1) \\ &= t\cdot v_1 \otimes t.v_1 = q^2 v_1\otimes v_1 \end{aligned}$$

$\Rightarrow v_1\otimes v_1$ is a highest weight vector with weight 2.

$$f\cdot(v_1\otimes v_1) = \Delta(f)(v_1\otimes v_1) = (f\otimes 1 + t\otimes f)(v_1\otimes v_1)$$
$$= f\cdot v_1\otimes v_1 + t\cdot v_1\otimes f\cdot v_1 = v_2\otimes v_1 + q\, v_1\otimes v_2$$

$$f^2\cdot(v_1\otimes v_1) = f\cdot\big(f\cdot(v_1\otimes v_1)\big)$$

$$= f\cdot\big(v_2\otimes v_1 + q\,v_1\otimes v_2\big) = (f\otimes 1 + t\otimes f)(v_2\otimes v_1 + q\,v_1\otimes v_2)$$

$$= q\,f\cdot v_1\otimes v_2 + t\cdot v_2\otimes f\cdot v_1 = q\,v_2\otimes v_2 + q^{-1}v_2\otimes v_2$$

$$= (q+q^{-1})(v_2\otimes v_2) = [2]\,v_2\otimes v_2$$

Define $e^{(m)} = \dfrac{e^m}{[m]}$, $f^{(m)} = \dfrac{f^m}{[m]}$.

$\Rightarrow e^{(1)} = e$, $f^{(1)} = f$

$\Rightarrow f^{(2)} = \dfrac{f^2}{[2]} \Rightarrow f^{(2)}\cdot v_1\otimes v_1 = v_2\otimes v_2$.

$$f^3\cdot(v_1\otimes v_1) = f\cdot\big(f^2\cdot(v_1\otimes v_1)\big) = f\cdot\big([2]\,v_2\otimes v_2\big)$$

$$= [2](f\otimes 1 + t\otimes f)(v_2\otimes v_2) = 0$$

$$\Rightarrow W_1 = \operatorname{span}_{\mathbb{C}(q)}\{v_1\otimes v_1,\ f^{(1)}\cdot(v_1\otimes v_1),\ f^{(2)}\cdot(v_1\otimes v_1)\}$$

$$= \operatorname{span}_{\mathbb{C}(q)}\{v_1\otimes v_1,\ v_2\otimes v_1 + q\,v_1\otimes v_2,\ v_2\otimes v_2\}$$

is an irred. $U_q(sl(2))$-submodule of $V_q\otimes V_q$.

Let $a\,v_2\otimes v_1 + b\,v_1\otimes v_2 \in V_q\otimes V_q$

$$e\cdot(av_2\otimes v_1 + bv_1\otimes v_2) = (e\otimes t^{-1} + 1\otimes e)(av_2\otimes v_1 + bv_1\otimes v_2)$$

$$= a q^{-1} v_1\otimes v_1 + b\,v_1\otimes v_1 = (aq^{-1}+b)(v_1\otimes v_1) = 0$$

$$\Rightarrow aq^{-1}+b = 0 \Rightarrow b = -aq^{-1} \Rightarrow a = -qb$$

$$a v_2 \otimes v_1 - a q^{-1} v_1 \otimes v_2 = \underbrace{(-aq^{-1})}_{\in\, \mathbb{C}(q)}(v_1 \otimes v_2 - q v_2 \otimes v_1)$$

$$\Rightarrow \quad e.(v_1 \otimes v_2 - q v_2 \otimes v_1) = 0$$

$$f.(v_1 \otimes v_2 - q v_2 \otimes v_1) = (f \otimes 1 + t \otimes f)(v_1 \otimes v_2 - q v_2 \otimes v_1)$$

$$= v_2 \otimes v_2 - q q^{-1} v_2 \otimes v_2 = v_2 \otimes v_2 - v_2 \otimes v_2 = 0.$$

$\Rightarrow W_2 = \operatorname{span}_{\mathbb{C}(q)} \{v_1 \otimes v_2 - q v_2 \otimes v_1\}$ is an irred. 1-dim'l $U_q(sl(2))$-submodule of $V_q \otimes V_q$

$$\Rightarrow V_q \otimes V_q = W_1 \oplus W_2 .$$

Setting $q = 0$ we get

W_1 basis is parametrized by the vectors $\{v_1 \otimes v_1,\ v_2 \otimes v_1,\ v_2 \otimes v_2\}$

and W_2 basis parametrized by the vector $v_1 \otimes v_2$.

$V_q \backslash V_q$: $v_1 \xrightarrow[1]{f} v_2$

$$\begin{array}{cccc} v_1 & & v_1 \otimes v_1 \rightarrow v_2 \otimes v_1 & V(2) \\ f \downarrow 1 & & \downarrow & \\ v_2 & v_1 \otimes v_2 & v_2 \otimes v_2 & \\ & V(0) & & \end{array}$$

Crystal graph of $V_q \otimes V_q$

1-dim'l $U_q(sl(2))$-module:

Let $V_q = \mathbb{C}(q)u$, $K = \mathbb{C}(q)$

is an one dim'l irred. $U_q(sl(2))$-module.

$t.u = cu$, $c \in K$.

$\Rightarrow v \in V_q$; $v = \alpha u$, $\alpha \in K$

$\Rightarrow t.v = t.(\alpha u) = \alpha(t.u) = c(\alpha u) = cv$

Set $e.u = v \in V_q$

$cv = c(e.u) = t.(e.u) = (te).u$

Recall $tet^{-1} = q^2 e \Rightarrow te = q^2 et$

$$\Rightarrow c(e.u) = q^2(et).u = q^2 e.(t.u)$$
$$= cq^2(e.u)$$

$\Rightarrow (c - cq^2)v = 0 \Rightarrow c(1-q^2)v = 0$

$\Rightarrow v = e.u = 0$ since $c \neq 0$, $q^2 \neq 1$

Consider $f.u = v \in V_q$

$cv = c(f.u) = t.(f.u) = (tf).u$

Recall $tft^{-1} = q^{-2}f \Rightarrow tf = q^{-2}ft$

$\Rightarrow cv = (q^{-2}ft).u = q^{-2}f.(t.u) = cq^{-2}f.u = cq^2 v$

$\Rightarrow c(1-q^{-2})v = 0 \Rightarrow v=0$ since $c \neq 0, q^2 \neq 1$

$\Rightarrow f.u = 0$

$$\Rightarrow \quad 0 = (e.f.u - f.e.u) = (ef-fe).u$$

$$= \left(\frac{t-t^{-1}}{q-q^{-1}}\right).u = \left(\frac{t.u - t^{-1}.u}{q-q^{-1}}\right)$$

$$= \frac{cu - c^{-1}u}{q-q^{-1}} = \left(\frac{c-c^{-1}}{q-q^{-1}}\right)u$$

$\Rightarrow \quad c = c^{-1}$ since $u \neq 0$

$\Rightarrow \quad c^2 = 1 \Rightarrow c = 1$ or -1

$\Rightarrow \exists$ (up to isom.) <u>two</u> irred. $U_q(sl(2))$ modules:

$V_q^+ = Ku^+ \quad (e.u^+ = 0 = f.u^+,\ t.u^+ = u^+)$

$V_q^- = Ku^- \quad (e.u^- = 0 = f.u^-,\ t.u^- = -u^-)$