$\mathfrak{g}$ Lie algebra / $\mathbb{C}$

$\mathcal{U}(\mathfrak{g})$ = univ. enveloping algebra of $\mathfrak{g}$.

Multiplication:

$$\mu : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$$
$$x \otimes y \longmapsto xy$$

μ is an alg. hom.

Comultiplication:

$$\Delta : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

$\forall\ x \in \mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$

Δ is an alg. hom

V $\mathfrak{g}$-module

$\Rightarrow$ V is a $\mathcal{U}(\mathfrak{g})$-module

and for V, W $\mathcal{U}(\mathfrak{g})$-modules

$V \otimes W$ is $\mathcal{U}(\mathfrak{g})$-module via

$$x.(v \otimes w) = \Delta(x)(v \otimes w) \quad \forall\ x \in \mathcal{U}(\mathfrak{g}),\ v \in V,\ w \in W$$

V $\mathfrak{g}$-module

$V^* = \{f : V \to \mathbb{C} \mid f \text{ linear}\}$ is a $\mathfrak{g}$-module via

$$(x \cdot f)(v) = -f(x \cdot v) \quad \forall\, x \in \mathfrak{g},\ f \in V^*,\ v \in V.$$

$\Rightarrow$ V $\mathcal{U}(\mathfrak{g})$-module

Let $x, y \in \mathfrak{g}$

$$((xy) \cdot f)(v) = (x \cdot \underbrace{(y \cdot f)})(v)$$

$$= -(y \cdot f)(x \cdot v) = (y \cdot f)((-x) \cdot v)$$

$$= -f(y \cdot (-x) \cdot v) = f((-y) \cdot ((-x) \cdot v))$$

$$= f(((-y)(-x)) \cdot v)$$

Define $S : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ to be an anti-homomorphism such that $\forall\, x \in \mathfrak{g}$

$$S(x) = -x$$

(i.e. $\forall\, x, y \in \mathfrak{g}$, $xy \in \mathcal{U}(\mathfrak{g})$ and $S(xy) = S(y)S(x)$.)

The anti-homomorphism $S : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is called an "antipode".

Recall $\forall\ x, y \in \mathfrak{g}$, $f \in V^*$, $v \in V$

$$\big((xy)\cdot f\big)(v) = f\big(((-y)(-x))\cdot v\big)$$

$$= f\big((S(y)S(x))\cdot v\big) = f\big(S(xy)\cdot v\big)$$

Thus for all $z \in \mathcal{U}(\mathfrak{g})$, $f \in V^*$, $v \in V$

V^* is a $\mathcal{U}(\mathfrak{g})$-module via

$$(z\cdot f)(v) = f\big(S(z)\cdot v\big).$$

Unit: $1 \in \mathcal{U}(\mathfrak{g})$ which can be defined as an alg. hom.

$$\eta : \mathbb{C} \longrightarrow \mathcal{U}(\mathfrak{g}), \quad (F = \mathbb{C})$$
$$1 \longmapsto 1$$

Co-unit:

$$\varepsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}$$

$\forall\ x \in \mathfrak{g}$, $\varepsilon(x) = 0$

is an alg. hom.

$(\mathcal{U}(\mathfrak{g}), \mu, \Delta, \eta, \varepsilon, S)$ is called an Hopf algebra.

Consider $\mathfrak{g} = sl(2, \mathbb{C}) = \langle e, f, h \rangle$

satisfying:

1) $[e, f] = h$
2) $[h, e] = 2e$
3) $[h, f] = -2f$

Then $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(sl(2))$ is an assoc. alg. with unit gen. by $\{f, h, e\}$ satisfying

1') $ef - fe = h$
2') $he - eh = 2e$
3') $hf - fh = 2f$

Recall that the quantum group $\mathcal{U}_q(\mathfrak{g}) = \mathcal{U}_q(sl(2))$ is an associative alg. over $\mathbb{C}(q)$ with unit gen. by $\{f, t = q^h, t^{-1}, e\}$

satisfying

1'') $ef - fe = \dfrac{t - t^{-1}}{q - q^{-1}}$

2'') $t e t^{-1} = q^{2} e$

3'') $t f t^{-1} = q^{-2} f$

Recall $\lim\limits_{q \to 1} U_q(sl(2)) = U(sl(2))$

$U_q(sl(2))$ is a Hopf algebra with the co-multiplication, co-unit and antipode defined as follows:

Co-multiplication : $\Delta : U_q(sl(2)) \to U_q(sl(2)) \otimes U_q(sl(2))$

defined by

- $\Delta(t^{\pm 1}) = t^{\pm 1} \otimes t^{\pm 1}$
- $\Delta(e) = e \otimes t^{-1} + 1 \otimes e$
- $\Delta(f) = f \otimes 1 + t \otimes f$

Δ is an alg. homomorphism.

Co-unit: $\varepsilon : U_q(sl(2)) \longrightarrow \mathbb{C}(q)$

- $\varepsilon(t^{\pm 1}) = 1, \quad \varepsilon(e) = 0 = \varepsilon(f)$

ε is an alg. homomorphism

Antipode: $S : U_q(sl(2)) \longrightarrow U_q(sl(2))$

- $S(t^{\pm 1}) = t^{\mp 1}, \quad S(e) = -et, \quad S(f) = -t^{-1}f$

S is an anti-homomorphism.

If V is an $U_q(sl(2))$-module and W is an $U_q(sl(2))$-module, then $V \otimes W$ is an $U_q(sl(2))$-module via $\forall\ x \in U_q(sl(2)),\ v \otimes w \in V \otimes W$

$$x.(v \otimes w) = \Delta(x)(v \otimes w)$$

Also V^* is an $U_q(sl(2))$-module via

$$(x.f)(v) = f(S(x).v)$$

$\forall\ x \in U_q(sl(2)),\ f \in V^*,\ v \in V.$

Recall that $\mathbb{C}^2$ is an irred $sl(2,\mathbb{C})$ module via

$$x \cdot v = xv \quad (\text{matrix mult.})$$

$\forall\ x \in sl(2),\ v \in \mathbb{C}^2$.

Hence $\mathbb{C}^2 \otimes \mathbb{C}^2$ is an $sl(2,\mathbb{C})$-module

$$\mathbb{C}^2 = V(1),$$

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = V(1) \otimes V(1)$$
$$= V(2) \oplus V(0)$$

$$V = \mathbb{C}^2 = \text{span}\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Define

$$V_q = \mathbb{C}(q)^2 = \text{span}_{\mathbb{C}(q)}\{v_1, v_2\}$$

is an irred. $U_q(sl(2))$-module.

Note: $t \cdot v_1 = q^h \cdot v_1 = q^{wt(h \cdot v_1)} v_1 = q v_1$

$$t \cdot v_2 = q^h \cdot v_2 = q^{wt(h \cdot v_2)} v_2 = q^{-1} v_2$$

$t^{-1} \cdot v_1 = q^{-1} v_1$, $t^{-1} \cdot v_2 = q v_2$, $e \cdot v_1 = 0$, $e \cdot v_2 = v_1$

$f \cdot v_1 = v_2$, $f \cdot v_2 = 0$.