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$\mathfrak{g}$ Lie algebra $/F$, $F = \mathbb{C}, \mathbb{Q}$

$$T^0\mathfrak{g} = F, \quad T^k\mathfrak{g} = \mathfrak{g}\otimes\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}$$

$$T\mathfrak{g} = \bigoplus_{k\geq 0} T^k\mathfrak{g} \quad \text{Tensor alg.}$$

$T\mathfrak{g}$ is a free associative algebra with unity.

$$\mathcal{U}(\mathfrak{g}) = T\mathfrak{g}/I$$

$$I = \langle x\otimes y - y\otimes x - [x,y] \mid x, y \in \mathfrak{g}\rangle$$

$\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and it is uniquely defined since it satisfies the universal property:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "\varphi"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dashed, "\exists!\,\psi \text{ assoc. alg. hom.}"] \\ & A \end{tikzcd}$$

associative alg.

V be $\mathfrak{g}$-module

$\Leftrightarrow \exists \quad \rho : \mathfrak{g} \longrightarrow gl(V)$ given by

$$\rho(x)v = x\cdot v$$

<u>PBW Thm</u>: $j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is 1-1.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "j"] \arrow[dr, "\rho"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dashed, "\exists!\,\psi"] \\ & \mathfrak{gl}(V) \end{tikzcd}$$

$\Rightarrow$ $\psi : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$ assoc. hom.

$\Rightarrow$ V is a $\mathcal{U}(\mathfrak{g})$-module.

Conversely, if we have V as $\mathcal{U}(\mathfrak{g})$-module, then $\exists$ assoc. alg. hom.

$$\psi : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$$

$$\psi\big|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \text{ is a}$$

Lie alg. hom.

$\Rightarrow$ V is a $\mathfrak{g}$-module

$\Longleftrightarrow$ V is a $\mathcal{U}(\mathfrak{g})$-module.

Recall:

<u>Thm</u>: Suppose $\{x_i \mid i \in I\}$ be a ordered basis for $\mathfrak{g}$. Then

$$\{x_{j_1} x_{j_2} \cdots x_{j_k} \mid k \geq 0,\ j_1 \leq j_2 \leq \cdots \leq j_k\}$$

is a basis for $\mathcal{U}(\mathfrak{g})$.

Definitions:

1) q-integers: $m \in \mathbb{Z}_{\geq 0}$

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$$

Ex: $[1] = 1$, $[0] = 0$

$$[2] = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}$$

$$[3] = \frac{q^3 - q^{-3}}{q - q^{-1}} = q^2 + 1 + q^{-2}$$

2) q-factorial:

$$[m]! = [m][m-1] \cdots [1]$$
$$[0]! = 1$$

3) q-binomial:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}$$

Ex: $\mathfrak{g} = sl(2, F)$

$\{f, h, e\}$ ordered basis for $\mathfrak{g}$.

Recall

$$[e, f] = h \,,\ [h, e] = 2e \,,\ [h, f] = -2f$$

$$\{x_{j_1} x_{j_2} \cdots x_{j_k} \mid k \geq 0,\ j_1 \leq j_1 \leq \cdots \leq j_k\}$$

basis for $\mathcal{U}(sl(2, \mathbb{C}))$, where $x_{j_i} \in \{f, h, e\}$

$$\Rightarrow \mathcal{U}(sl(2, F)) = \operatorname{span}\{f^m h^k e^p \mid m, k, p \in \mathbb{Z}_{\geq 0}\}.$$

In deed, we can define $\mathcal{U}(sl(2, F))$ via generators and relations as follows:

$\mathcal{U}(\mathfrak{g})$ is the associative algebra with unity gen. by $\{f, h, e\}$ satisfying the relations:

(1) $ef - fe = h$

(2) $he - eh = 2e$

(3) $hf - fh = -2f$.

Let q be an indeterminant such that $q^m \neq 1$ for any m. (i.e. q is generic.

Quantum group $U_q(sl(2))$:

It is an associative algebra over $F(q)$ with unity generated by $\{f, t = q^h, t^{-1}, e\}$ satisfying the relations:

(1) $ef - fe = \dfrac{t - t^{-1}}{q - q^{-1}}$

(2) $tet^{-1} = q^2 e$

(3) $tft^{-1} = q^{-2} f$

Remark: $\lim_{q \to 1} U_q(sl(2)) = U(sl(2))$

(1) $(ef - fe)(q - q^{-1}) = q^h - q^{-h}$

Take $q \to 1$:

differentiating w.r.to q

$$(ef - fe)(1 + q^{-2}) = h q^{h-1} + h q^{-h-1}$$

Set $q = 1$,

$$(ef - fe)(2) = 2h$$

$$\Rightarrow \quad ef - fe = h.$$

(2) $tet^{-1} = q^2 e$

$\Rightarrow q^h e q^{-h} = q^2 e$

Differentiate w.r.to q:

$$h q^{h-1} e q^{-h} + q^h e (-h) q^{-h-1} = 2qe$$

Set $q=1$:

$$he - eh = 2e.$$

(3) exercise.

V, W be $\mathfrak{g}$-modules

Recall that $V \otimes W$ is a $\mathfrak{g}$-module via

$(*)$ $\quad x.(v \otimes w) = (x.v) \otimes w + v \otimes x.w$

for all $x \in \mathfrak{g}$, $v \in V$, $w \in W$

Let us define an algebra hom.

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall x \in \mathfrak{g}$$

Δ is called a comultiplication.

For $x, y \in \mathfrak{g}$

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y)\\ &= (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y)\\ &= xy\otimes 1 + x\otimes y + y\otimes x + 1\otimes xy\end{aligned}$$

Remark:

$$x\cdot(v\otimes w) = (x\cdot v)\otimes w + v\otimes(x\cdot w)$$

can be written as

$$x\cdot(v\otimes w) = \Delta(x)(v\otimes w)$$

If V and W are $\mathcal{U}(\mathfrak{g})$-modules we can define the $\mathcal{U}(\mathfrak{g})$-module action on $V\otimes W$ by

$$z\cdot(v\otimes w) = \Delta(z)(v\otimes w)$$

for all $z\in\mathcal{U}(\mathfrak{g})$, $v\in V$, $w\in W$.