Exercises

Nil Hecke

Consider $V = \mathbb{C}[x_1, x_2, \ldots, x_m]$

The operator x_t acts by left mult.
Operators $\sigma \in \mathfrak{S}_m$ act by permutation
so $\sigma(x_t) = x_{\sigma(t)}$
but if $f \in V$, $\sigma f(x_1 \ldots x_m) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)})$.

Exercise [Verify!]

Define operators ∂_r on V by
$$\partial_r f = \frac{f - s_r f}{x_r - x_{r+1}}$$

Exercise [Verify $\partial_r f$ is a polynomial].

Exercise Verify ∂_r satisfy the braid relations
What is ∂_r^2 ?
How do ∂_r and x_r interact ?

We say $NilHt_m = \langle \partial_r \ 1 \le r < m, \ x_t \ 1 \le t \le m \rangle /$ relations

~~Let M be a fin dim repr of NilHt_m.~~
$\deg(x_t) = 2$, $\deg(\partial_r) = -2$

Exercises

Nil Hecke

continued.

Let $m=2$

Let M be an irrep of $NilH_2$.

Exercise Can $\dim M = 1$?

If so $M = \mathrm{span}\{v\}$ and $x_1 v = av$ $x_2 v = bv$ $\partial_1 v = cv$

and what can you say about a, b, c?

Let $L(a)$ be the 1-dim rep of $NilH_1 = \mathbb{C}[x]$ on which $x-a$ vanishes.

Note $L(a) \boxtimes L(b)$ denotes the 1-dim rep of $NilH_1 \otimes NilH_1 \cong \mathbb{C}[x_1, x_2]$ on which $x_1 - a$, $x_2 - b$ vanish.

Exercise Compute a basis of $\mathrm{Ind}_{NilH_1 \otimes NilH_1}^{NilH_2} L(a) \boxtimes L(b)$

and the explicit action of generators of $NilH_2$ on this basis.

In the case $a = b = 0$, you should ~~recognize~~ observe this agrees with $V / V_+^{S_2}$

Exercise What is its graded character?

Compare to quantum shuffle...

Exercises

Nil Hecke

Note symmetric polynomials $V^{S_m} \subseteq V$ generate an $NilH_m$-invariant subspace $V_+^{S_m}$
$= \mathbb{C}[x_1 \dots x_m]\langle e_1, e_2, \dots, e_m \rangle$

Exercise [Verify]

so $V/V_+^{S_m}$ is an $m!$-dim repr of $NilH_m$,

on which $\mathbb{C}[x_1 \dots x_m]^{S_m} = Z(NilH_m)$ acts trivially (in fact, the unique such irrep).

Exercise For $m=2$, write out the action of the generators explicitly on some basis.

$NilH_{m-1} \subseteq NilH_m$, making $NilH_m$ into a free right (or left) $NilH_{m-1}$ module.

Exercise Write down a basis.

Let's call $V/V_+^{S_m} = L(0^m)$.
Then we can restrict $\mathrm{Res}^m_{m-1} L(0^m)$ to a $NilH_{m-1}$-module.

Exercise What is its structure as a $NilH_{m-1}$-module?

Hint: Think about the uniqueness of $L(0^{m-1})$.
Does your guess work for $m=2$?

Hint: [m]$_q$

Exercises

up to grading shift

<u>Claim</u> $\quad \mathrm{soc}\ \mathrm{Res}^{m}_{m-1} L(0^m) = L(0^{m-1})$

and in particular is irreducible.
(soc stands for "socle" which is the largest semisimple submodule and coincides with the sum of all simple submodules;
dually, the cosocle is the largest semisimple quotient.)

This phenomenon is one big ingredient in defining our crystal operators.

dually:
<u>Claim</u> $\quad \mathrm{cosoc}\ \mathrm{Ind}^{m+1}_{m,1} L(0^m) \boxtimes (1) = L(0^{m+1})$

~~If we declare, for an irrep M of Nil~~

Recall if $A \subseteq B$ are nice (sub) algebras
$\mathrm{Ind}_A^B M = B \otimes_A M$

Exercises

KLR-algebra for $\mathfrak{sl}_2$

Dynkin $\bullet$

$|I| = 1$, let's call $I = \{i\}$

The handout/slide has the presentation of $R(\nu)$.

Exercise: Write out the generators for

(a) $\nu = \phi$ (b) $\nu = i$ (c) $\nu = mi$

(aka $\nu = 0$; $\nu = \alpha_i$; $\nu = m\alpha_i$)

Let Pol_m = subalgebra of $R(mi)$ generated by $x_1, \dots, x_m$

Exercise: What are all the irreps of Pol_m ?

(a) for $m=1$; (b) for general m ?

Consider $M = \mathrm{Ind}_{Pol_m}^{R(mi)} L(i) \boxtimes \cdots \boxtimes L(i)$

Exercise: What is $\dim M$?

Ex: Write down a basis of M.

Ex: Do the x_t act triangularly on this basis ?

Ex: For $m=2$ write out explicitly how generators act.

In that case, what is $\mathrm{Res}_{Pol_m} M$?

KLR for $\mathfrak{sl}_2$

Let M be an irred $R(mi)$-module. Set $\mathrm{wt}\, M = -mi$
$= -m\alpha_i$

Declare
$$\tilde{f}_i M = \mathrm{cosoc}\ \mathrm{Ind}_m^{m+1} M \boxtimes L(i) .$$

Build a crystal

$$L(\emptyset) \xrightarrow{i} L(i) \xrightarrow{i} L(i^2) \xrightarrow{i} L(i^3) \xrightarrow{i} \cdots$$

<u>Exercise</u> What crystal is this?

<u>Grothendieck groups</u>

If $\mathrm{Rep}\, R$ = category of fin dim R-modules
$G_0(\mathrm{Rep}\, R)$ = $\mathbb{Z}$-module generated by
$[N]$, N = fin dim R-mod
and relations $0 \to A \to B \to C \to 0$ exact $\Rightarrow$
$[B] = [A] + [C]$.

If R is graded, we get a $\mathbb{Z}[q, q^{-1}]$-module

In the above our crystal came from an underlying space $\bigoplus_m G_0(\mathrm{Rep}\, R(mi))$