Crystal theory
and
LLTA theory

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Southeastern
Lie Theory Workshop
October 9, 2009

Crystal bases are Bases at $q = 0$

Representation theory

Crystal Bases

Combinatorics

Representation theory of Affine Heck algebras

Crystal bases of affine quantum groups

§1. Definition of $U_q(\mathfrak{g})$

Data

I : an index set of simple roots

P: a free $\mathbb{Z}$-module, called weight lattice

$\alpha_i \in P$: simple roots $(i \in I)$

$h_i \in P^* := \mathrm{Hom}(P, \mathbb{Z})$: simple coroots

(,) : an inner product on P

satisfying $\langle h_i, \lambda \rangle = \dfrac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad (\forall \lambda \in P)$

$U_q(\mathfrak{g})$ is the algebra generated by symbols e_i, f_i, q^h $(h \in P^*)$ with defining relations:

$$q^h = 1 \text{ for } h = 0, \qquad q^{h+h'} = q^h q^{h'},$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i,$$

$$q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$$

with $q_i := q^{(\alpha_i,\alpha_i)/2}$, $t_i := q_i^{h_i} = q^{\frac{(\alpha_i,\alpha_i)}{2} h_i}$,

$$\left\{ \begin{array}{ll} \sum_{n=0}^{b} (-1)^n e_i^{(n)} e_j e_i^{(b-n)} = 0 & \\ & \text{for } i \neq j \\ \sum_{n=0}^{b} (-1)^n f_i^{(n)} f_j f_i^{(b-n)} = 0 & \end{array} \right.$$

$b = 1 - \langle h_i, \alpha_j \rangle$ (q-Serre relations)

$$[n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := [1]_i \cdots [n]_i,$$

$$e_i^{(n)} := e_i^n / [n]_i!, \ f_i^{(n)} := f_i^n / [n]_i!.$$

$U_q(\mathfrak{g}) \supset U_q(\mathfrak{sl}_2)_i = \langle e_i, f_i, t_i^{\pm} \rangle$

quantized $\mathfrak{sl}_2$

$U_q(\mathfrak{g}) \supset U_q(\mathfrak{sl}_2)_i = \langle e_i, f_i, t_i^{\pm} \rangle$ quantized $\mathfrak{sl}_2$

Def. A $U_q(\mathfrak{g})$-module M is integrable if

- $M = \bigoplus_{\lambda \in P} M_\lambda$.

  $$M_\lambda = \left\{ u \in M \,;\, q^h u = q^{\langle h, \lambda \rangle} u \right\}.$$
- For any $i \in I$, M is a union of finite-dimensional $U_q(\mathfrak{sl}_2)_i$-modules.

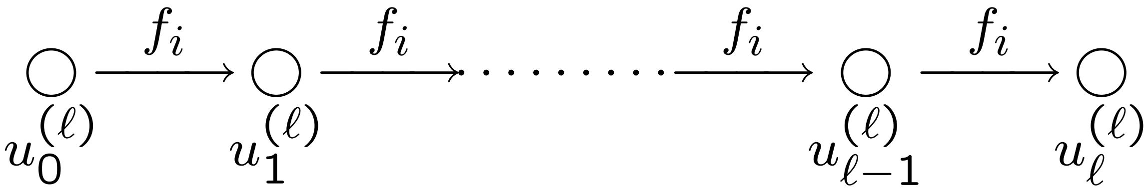
$\Rightarrow$ $M \simeq \bigoplus_\nu V_{\ell_\nu}$ as a $U_q(\mathfrak{sl}_2)_i$-module

$$V_\ell = \langle u_0^{(\ell)}, u_1^{(\ell)}, \ldots, u_\ell^{(\ell)} \rangle$$

$$t_i u_\nu^{(\ell)} = q_i^{\ell - 2\nu} u_\nu^{(\ell)},$$

$$f_i u_\nu^{(\ell)} = [\nu + 1]_i \; u_{\nu+1}^{(\ell)},$$

$$e_i u_\nu^{(\ell)} = [\ell - \nu + 1]_i \; u_{\nu-1}^{(\ell)}.$$



§2. Crystal bases

$K = \mathbb{C}(q)$
$\cup$
$\mathsf{A}_0 = \{f \in K; f \text{ is regular at } q = 0\}$

V : a K-vector space

Def. (L, B) is a local basis of V at $q = 0$ if

- L is a free A_0-submodule of V such that $V = K \underset{\mathsf{A}_0}{\otimes} L$.
- $B \subset L/qL$ is a basis as a $\mathbb{C}$-vector space.

$\dim V = \# B$

M : an integrable $U_q(\mathfrak{g})$-module

Def. (L, B) is a crystal basis of M if

- (L, B) is a local basis of M at $q = 0$.
- For each $i \in I$, we can find
  a $U_q(\mathfrak{sl}_2)_i$-linear isomorphism

$$\begin{array}{ccc} M & \simeq & \bigoplus_\nu V_{\ell_\nu} \\ (L, B) & \leftrightarrow & \{u_k^{(\ell_\nu)}; 0 \le k \le \ell_\nu\}. \end{array}$$

Define the modified root operators

$\tilde{e}_i u = \sum_n f_i^{(n-1)} u_n, \quad \tilde{f}_i u = \sum_n f_i^{(n+1)} u_n$

where $u = \sum_n f_i^{(n)} u_n$ with $e_i u_n = 0$

$\tilde{e}_i,\ \tilde{f}_i : B \to B \bigsqcup \{0\}$

$$\tilde{f}_i(u_k^{(\ell)}) = \begin{cases} u_{k+1}^{(\ell)} & \text{if } 0 \le k < \ell \\ 0 & \text{if } k = \ell \end{cases}$$

$$\tilde{e}_i(u_k^{(\ell)}) = \begin{cases} u_{k-1}^{(\ell)} & \text{if } 0 < k \le \ell \\ 0 & \text{if } k = 0. \end{cases}$$

$$b \xrightarrow{i} \tilde{f}_i b$$

i-string

$$\overset{u_0^{(\ell)}}{\bigcirc} \underbrace{\xrightarrow{i} \cdots\cdots \xrightarrow{i}}_{\varepsilon_i(b)} \underset{b}{\bigcirc} \underbrace{\xrightarrow{i} \cdots\cdots \xrightarrow{i}}_{\varphi_i(b)} \overset{u_\ell^{(\ell)}}{\bigcirc}$$

$$\langle h_i, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$$

For any i, B is a disjoint union of i-strings.

$\lambda \in P^+ := \{\lambda \in P : \langle h_i, \lambda\rangle \geq 0\}$

(dominant integral weight)

$$\begin{aligned} V(\lambda) &:= \text{the irreducible } U_q(\mathfrak{g})\text{-module} \\ &\quad\ \text{with highest weight } \lambda \\ &= U_q(\mathfrak{g})u_\lambda \text{ with defining relations} \end{aligned}$$

$$\begin{cases} (\text{weight } \lambda) \quad q^h u_\lambda = q^{\langle h,\lambda\rangle} u_\lambda \ (\forall h \in P^*) \\ (\text{highest weight}) \quad e_i u_\lambda = 0 \\ f_i^{\langle h_i,\lambda\rangle+1} u_\lambda = 0 \end{cases}$$

$\mathrm{O}_{\mathrm{int}}$:= the category of integrable $U_q(\mathfrak{g})$-modules such that $\dim U_q^+(\mathfrak{g})u < \infty$ for any $u \in M$

$\mathrm{O}_{\mathrm{int}}$ is semisimple $\quad \mathrm{O}_{\mathrm{int}} = \{M ; M \simeq \bigoplus_\nu V(\lambda_\nu)\}$

$$U_q^+(\mathfrak{g}) := \langle e_i ; i \in I\rangle \subset U_q(\mathfrak{g})$$

Th. (Existence)

- $V(\lambda)$ has a unique crystal basis $(L(\lambda), B(\lambda))$ such that $B(\lambda)_\lambda = \{u_\lambda\}$.

- $\{b \in B(\lambda); \tilde{e}_i(b) = 0$ for $\forall i \in I\} = \{u_\lambda\}$.

- $B(\lambda)$ is connected.

Th. (Uniqueness)

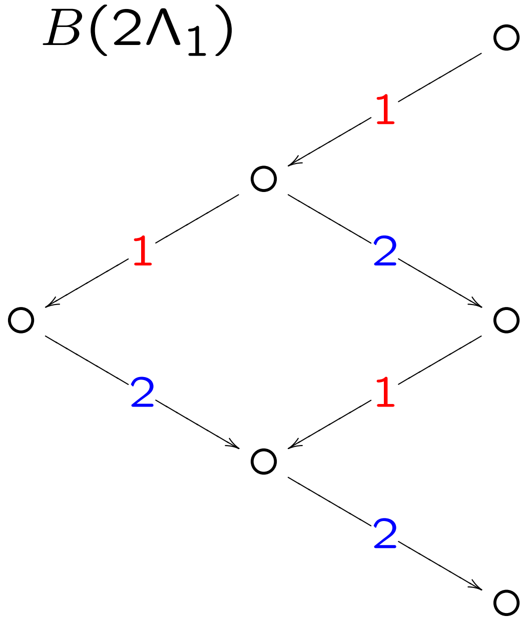
$M \in \mathrm{O}_{\mathrm{int}}$

(L, B): a crystal basis of M

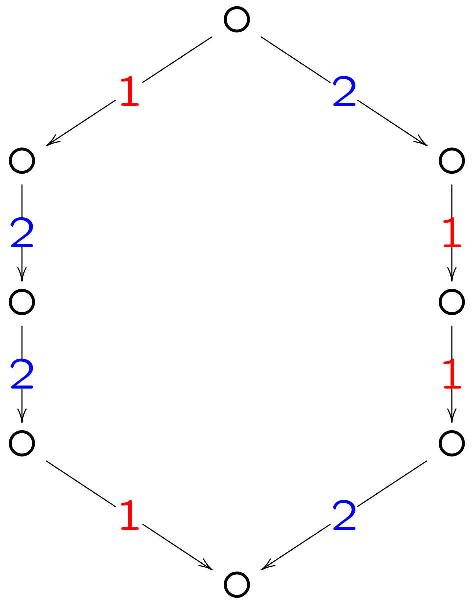
$\Rightarrow$

$$\begin{array}{ccc} M & \simeq & \bigoplus_\nu V(\lambda_\nu) \\ (L, B) & \leftrightarrow & \bigoplus_\nu (L(\lambda_\nu), B(\lambda_\nu)). \end{array}$$

$\mathfrak{g} = \mathfrak{sl}_3$ $\quad B(2\Lambda_1)$



the adjoint representation $B(\Lambda_1 + \Lambda_2)$



Th. (Stability under $\otimes$)

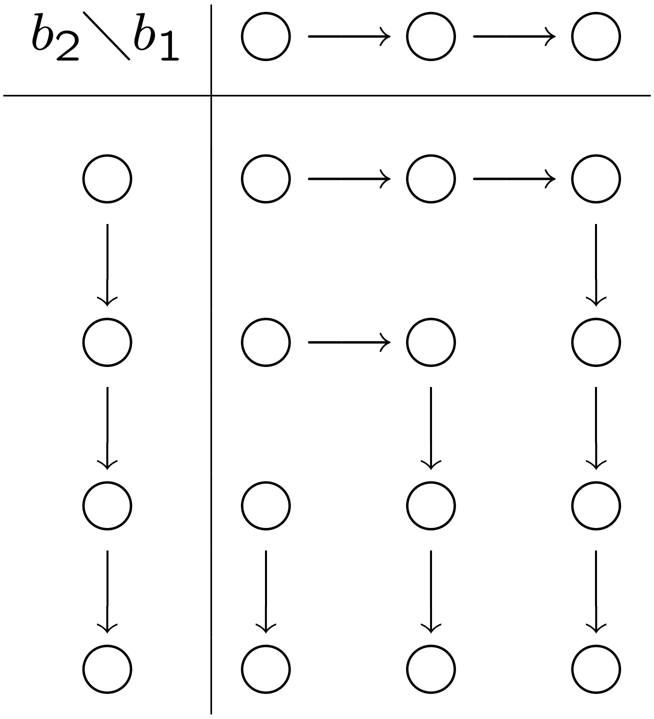
M_ν: integrable $U_q(\mathfrak{g})$-modules ($\nu = 1, 2$)

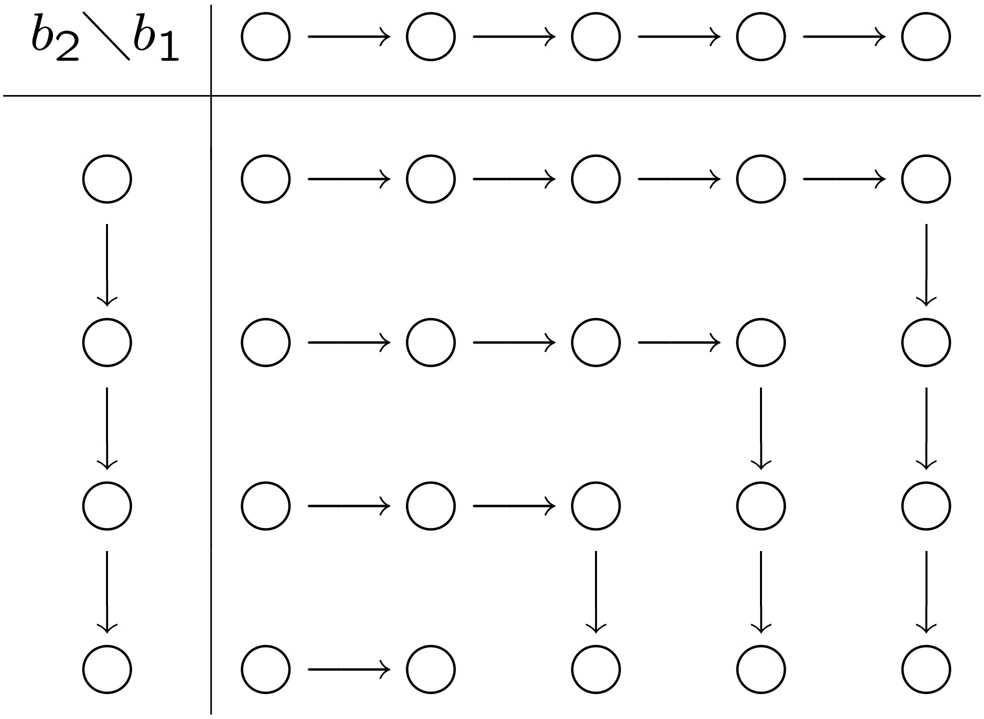
(L_ν, B_ν): a crystal basis of M_ν

$\Rightarrow$

- $(L_1 \otimes L_2, B_1 \otimes B_2)$
  is a crystal basis of $M_1 \otimes M_2$.
- $$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i(b_2) & \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

  $$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$


$b_2 \diagdown b_1$


$b_2 \diagdown b_1$

Cor. $M \in \mathrm{O}_{\mathrm{int}}$

(L, B): a crystal basis of M

$$\Rightarrow M \simeq \bigoplus_{\substack{b \in B \\ \varepsilon_i(b)=0 \ \forall i}} V(\mathrm{wt}(b)).$$

Cor. (Decomposition theorem)

$$\boxed{V(\lambda) \otimes V(\mu) \simeq \bigoplus_{\substack{b \in B(\mu) \\ \varepsilon_i(b) \leq \langle h_i, \lambda \rangle \ \forall i}} V(\lambda + \mathrm{wt}(b)).}$$

$$\because \quad \varepsilon_i(b_1 \otimes b_2) = 0$$
$$\iff \varepsilon_i(b_1) = 0 \quad \text{and} \quad \varepsilon_i(b_2) \leq \varphi_i(b_1)$$
$$\iff b_1 = u_\lambda \quad \text{and} \quad \varepsilon_i(b_2) \leq \langle h_i, \lambda \rangle$$

Example $\mathfrak{g} = \mathfrak{sl}_n$

$I = \{1, 2, \ldots, n-1\}$,

$P = \bigoplus_{k=1}^{n} \mathbb{Z}\epsilon_k$,

$\alpha_k = \epsilon_k - \epsilon_{k+1}$,

$\Lambda_k = \epsilon_1 + \cdots + \epsilon_k$: the fundamental weights

$B(\Lambda_1)$: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots\cdots \xrightarrow{n-2} \boxed{\text{n-1}} \xrightarrow{n-1} \boxed{\text{n}}$

Th.

$B(\lambda) =$ the set of semi-standard Young tableaux with shape λ.

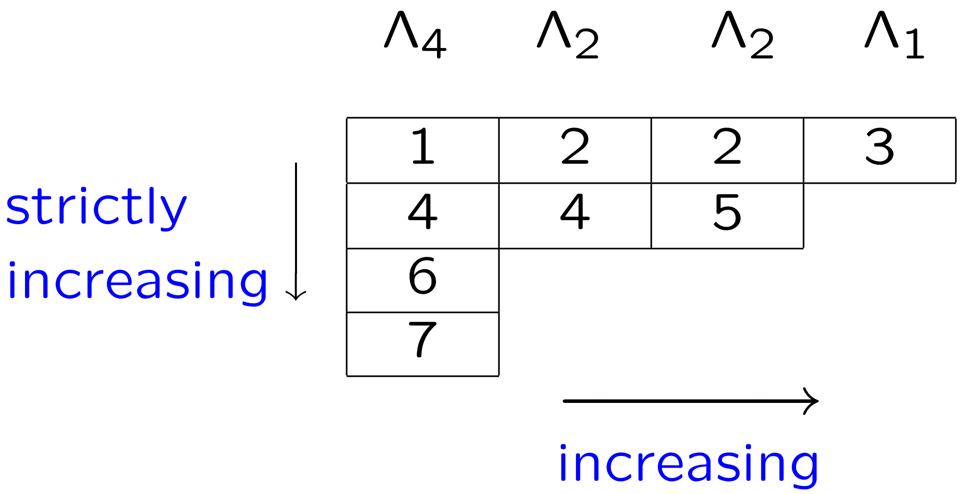
$B(\lambda) \subset B(\Lambda_1)^{\otimes N}$,

Example $\lambda = \Lambda_1 + 2\Lambda_2 + \Lambda_4$

$$V(\lambda) \subset V(\Lambda_1) \otimes V(\Lambda_2) \otimes V(\Lambda_2) \otimes V(\Lambda_4)$$

$$B(\lambda) \subset B(\Lambda_1) \otimes B(\Lambda_2) \otimes B(\Lambda_2) \otimes B(\Lambda_4)$$

$$B(\Lambda_i) \subset B(\Lambda_1)^{\otimes i}$$



$$\begin{array}{c|c|c|c|}
\Lambda_4 & \Lambda_2 & \Lambda_2 & \Lambda_1 \\
\hline
1 & 2 & 2 & 3 \\
\hline
4 & 4 & 5 \\
\cline{1-3}
6 \\
\cline{1-1}
7 \\
\cline{1-1}
\end{array}$$

strictly increasing $\downarrow$

$\longrightarrow$ increasing

$$\boxed{3} \otimes \boxed{2} \otimes \boxed{5} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{4} \otimes \boxed{6} \otimes \boxed{7}$$

in $B(\Lambda_1)^{\otimes 9}$

Crystal basis of $U_q^-(\mathfrak{g})$

$$\begin{tikzcd}
U_q^-(\mathfrak{g}) \arrow[rr, two heads] \arrow[dr, "\fallingdotseq"'] & & V(\lambda) \\
& \varprojlim_{\lambda} V(\lambda) \arrow[ur] &
\end{tikzcd}$$

$e_i' \in \mathrm{End}_K(U_q^-(\mathfrak{g}))$

$e_i'(ab) = e_i'(a) \cdot b + \mathrm{Ad}(t_i)a \cdot e_i'b, \quad e_i'(f_j) = \delta_{ij}$

$$e_i' \cdot f_i = q_i^{-2} f_i \cdot e_i' + 1 \quad (q\text{-boson})$$

$e_i' \cdot f_j = q^{-(\alpha_i, \alpha_j)} f_j \cdot e_i' \qquad (i \neq j)$

$\mathcal{B}(\mathfrak{g})$: the K-subalgebra generated by the f_i's and e_i''s of $\mathrm{End}_K(U_q^-(\mathfrak{g}))$

$\mathfrak{g}_+$: the Lie algebra generated by e_i's

G_+: the group with $\mathfrak{g}_+$ as its Lie algebra,

$\mathbb{C}[G_+]$: the ring of functions on G_+

$\mathcal{D}[G_+]$: the ring of differential operators on G_+

$$\begin{tikzcd}
\mathcal{D}[G_+] & \text{acts on} & \mathbb{C}[G_+] \arrow[r, leftrightarrow, bend left, "\text{duality}"] & U(\mathfrak{g}_+) \\
\mathcal{B}(\mathfrak{g}) & \text{acts on} \arrow[u, Rightarrow, "q\to 1"'] & U_q^-(\mathfrak{g}) \arrow[r, leftrightarrow, bend left, "\text{duality}"] & U_q^-(\mathfrak{g})
\end{tikzcd}$$

Any $u \in U_q^-(\mathfrak{g})$ is written as

$$u = \sum_n f_i^{(n)} u_n \quad \text{where } e_i' u_n = 0$$

$$\tilde{f}_i(u) = \sum_n f_i^{(n+1)} u_n, \quad \tilde{e}_i(u) = \sum_n f_i^{(n-1)} u_n$$

$$\begin{aligned} L(U_q^-(\mathfrak{g})) &= \sum \mathsf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} 1, \\ B(U_q^-(\mathfrak{g})) &= \Big\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} 1 \text{ mod } qL(U_q^-(\mathfrak{g}))\,; \\ &\quad\; l \geq 0,\, i_1, \ldots i_l \in I \} \\ &\quad \subset L(U_q^-(\mathfrak{g}))/qL(U_q^-(\mathfrak{g})) \end{aligned}$$

Th. $(L(U_q^-(\mathfrak{g})), B(U_q^-(\mathfrak{g})))$ is a local basis of $U_q^-(\mathfrak{g})$, called the crystal basis of $U_q^-(\mathfrak{g})$.

Global bases

$$K := \mathbb{C}(q) \supset \begin{array}{lcl} \mathsf{A}_0 & := & \{\text{functions regular at } q = 0\} \\ \mathsf{A}_\infty & := & \{\text{functions regular at } q = \infty\} \\ \mathsf{A} & := & \mathbb{C}[q, q^{-1}] \end{array}$$

V: a K-vector space

$L_0 \subset V$: an A_0-module such that $V = K \otimes L_0$

$L_\infty \subset V$: an A_∞-module such that $V = K \otimes L_\infty$

$V_\mathbb{Z} \subset V$: an A-module

Define the vector bundle $\mathscr{V}$ on $\mathbf{P}^1$:

V is the space of meromorphic sections of $\mathscr{V}$,
L_0 is the germ of $\mathscr{V}$ at $q = 0$,
L_∞ is the germ of $\mathscr{V}$ at $q = \infty$,
$V_\mathbb{Z}$ is the space of sections of $\mathscr{V}$ over $\mathbf{P}^1 \setminus \{0, \infty\}$.

$E := V_\mathbb{Z} \cap L_0 \cap L_\infty$ the space of global sections

Prop. The following conditions are equivalent.

(i) $E \xrightarrow{\sim} L_0/qL_0$.

(ii) $(V_{\mathbb{Z}} \cap qL_0) \oplus (V_{\mathbb{Z}} \cap L_\infty) \xrightarrow{\sim} V_{\mathbb{Z}}$.

(iii)
$$\begin{aligned} K \otimes_{\mathbb{C}} E &\xrightarrow{\sim} V, \\ \mathsf{A} \otimes_{\mathbb{C}} E &\xrightarrow{\sim} V_{\mathbb{Z}}, \\ \mathsf{A}_0 \otimes_{\mathbb{C}} E &\xrightarrow{\sim} L_0 \\ \mathsf{A}_\infty \otimes_{\mathbb{C}} E &\xrightarrow{\sim} L_\infty. \end{aligned}$$

In such a case we say $(V_{\mathbb{Z}}, L_0, L_\infty)$ is balanced.

$G : L_0/qL_0 \xrightarrow{\sim} E$.

$B \subset L_0/qL_0$ is a basis

$\Rightarrow G(B)$ is a basis of V, $V_{\mathbb{Z}}$ and L_0, L_∞

$G(B)$ is called a global basis

$- : K \to K \ (q \mapsto q^{-1})$

$- : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$

a unique map such that

$\overline{au} = \overline{a} \cdot \overline{u} \ (a \in K,\ u \in U_q^-(\mathfrak{g}))$,

$\overline{e_i u} = e_i \overline{u}$ and $\overline{1} = 1$

$U_q^-(\mathfrak{g})_{\mathbb{Z}}$ is the largest A-submodule of $U_q^-(\mathfrak{g})$ such that $e_i^{(n)} U_q^-(\mathfrak{g})_{\mathbb{Z}} \subset U_q^-(\mathfrak{g})_{\mathbb{Z}}$ and $U_q^-(\mathfrak{g})_{\mathbb{Z}} \cap K \subset$ A

Th. $(U_q^-(\mathfrak{g})_{\mathbb{Z}}, L(U_q^-(\mathfrak{g})), \overline{L(U_q^-(\mathfrak{g}))})$ is balanced.

$G : L(U_q^-(\mathfrak{g}))/qL(U_q^-(\mathfrak{g})) \to U_q^-(\mathfrak{g})$

$G(B(U_q^-(\mathfrak{g})))$ is a basis of $U_q^-(\mathfrak{g})$ (upper global basis)

Lascoux-Leclerc-Thibon-Ariki Theory

on

Affine Hecke algebra H_n^A of type A and crystal bases

H_n^A is a q-analogue of the affine Weyl group ring

$p \in \mathbb{C}^*$ (usually written as q)

Def. H_n^A is a $\mathbb{C}$-algebra generated by $T_1, \dots T_{n-1}$ and invertible elements $X_1, \dots X_n$
with the defining relation:

- $(T_i - p)(T_i + p^{-1}) = 0$
- braid relations $T_iT_j = T_jT_i \ (|i-j| > 1)$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$
- X_i's commute each other,
- $T_iX_iT_i = X_{i+1}, \quad T_iX_j = X_jT_i \ (j \neq i, i+1)$

$I \subset \mathbb{C}^*$

$\mathsf{Mod}_I(\mathsf{H}_n^A)$:the category of finite-dimensional H_n^A-modules such that all the eigenvalues of X_i are in I

K_n^A := the Grothendieck group of the abelian category $\mathsf{Mod}_I(\mathsf{H}_n^A)$

= generated by $[M]$ ($M \in \mathsf{Mod}_I(\mathsf{H}_n^A)$) with
$[M] = [M'] + [M'']$ for exact sequences

$$0 \to M' \to M \to M'' \to 0$$

It has a basis $\Big\{[S]; [S]$ is a simple H_n^A-modules$\Big\}$

$\mathsf{K}^A = \bigoplus_n \mathsf{K}_n^A$ has a structure of a commutative ring.

H_{m+n}^A is generated by

$$\begin{array}{c} \overbrace{T_1, T_2, \ldots, T_{m-1}}, \ T_m \ , \overbrace{T_{m+1}, \ldots, T_{m+n-1}} \\ \underbrace{X_1, X_2, \ldots\ldots\ldots, X_m}_{\mathsf{H}_m^A}, \ \underbrace{X_{m+1}, \ldots\ldots\ldots, X_{m+n}}_{\mathsf{H}_n^A} \end{array}$$

$$\mathsf{H}_m^A \otimes \mathsf{H}_n^A \subset \mathsf{H}_{m+n}^A$$

For an H_m^A-module M and an H_n^A-module N

$$\begin{array}{ccccc} [M] & \cdot & [N] & = & \left[\mathrm{Ind}_{\mathsf{H}_m^A \otimes \mathsf{H}_n^A}^{\mathsf{H}_{m+n}^A}(M \otimes N)\right] \\ \Cap & & \Cap & & \Cap \\ \mathsf{K}_m^A & & \mathsf{K}_n^A & & \mathsf{K}_{m+n}^A \end{array}$$

$$\mathrm{Ind}_{\mathsf{H}_m^A \otimes \mathsf{H}_n^A}^{\mathsf{H}_{m+n}^A}(M \otimes N) = \mathsf{H}_{m+n}^A \underset{\mathsf{H}_m^A \otimes \mathsf{H}_n^A}{\otimes} (M \otimes N)$$

$a \in I \subset \mathbb{C}^*$,

$e_a : \mathsf{K}_n^A \to \mathsf{K}_{n-1}^A$

$$\begin{array}{ccc} \mathrm{Mod}_I(\mathsf{H}_n^A) & & \mathrm{Mod}_I(\mathsf{H}_{n-1}^A) \\ \Cup & & \Cup \\ M & \longmapsto & \left\{\begin{array}{l} \text{the generalized eigenspaces of} \\ X_n \text{ with eigenvalue } a \end{array}\right\} \end{array}$$

$$(a,b) = \begin{cases} 2 & \text{if } a = b \\ -1 & \text{if } a = p^{\pm 2} b \\ 0 & \text{otherwise} \end{cases}$$

Dynkin diagram

$$\cdots - \boxed{p^{-4}a} - \boxed{p^{-2}a} - \boxed{a} - \boxed{p^{2}a} - \boxed{p^{4}a} - \cdots$$

$\mathfrak{g}$ is a union of

$\mathfrak{gl}_\infty$ if p is not a root of unity,

$A_\ell^{(1)}$ if p^2 is a primitive ℓ-th root of unity

G_+: the group with the Lie algebra

$$\mathfrak{g}_+ = \{\text{generated by the } e_i\text{'s}\}$$

Th.

(i) $\mathbb{C} \otimes_{\mathbb{Z}} \mathsf{K}^A \simeq \mathbb{C}[G_+]$

$\simeq$the specialization of $U_q^-(\mathfrak{g})$ at $q = 1$,

(ii) the irreducible modules correspond to the upper global basis (at q=1).

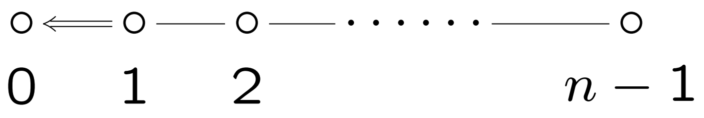
$$\begin{tikzcd}
\mathcal{D}(G_+) & \text{acts on} & \mathbb{C}[G_+] & \overset{\simeq}{} \mathbb{C} \otimes_{\mathbb{Z}} \mathsf{K}^A \supset \{\text{simple modules}\} \\
\mathcal{B}(\mathfrak{g}) & \text{acts on} & U_q^-(\mathfrak{g})_{\mathbb{Z}} \arrow[u, Rightarrow, "q\to 1"'] & \supset \text{upper global basis}
\end{tikzcd}$$

You see a new symmetry when considering all the H_n^A's at onece.

Conjectural L-L-T-A Theory for affine Hecke algebra H_n^B of type B

(with Naoya Enomoto)

Dynkin diagram of type B_n



$p_0, p \in \mathbb{C}^*$

H_n^B is a $\mathbb{C}$-algebra generated by $T_0, T_1, \dots T_{n-1}$ and invertible elements $X_1, \dots X_n$

with the defining relation:

- $(T_i - p_i)(T_i + p_i^{-1}) = 0$ $(p_1 = \cdots = p_{n-1} = p)$
- braid relations $T_iT_j = T_jT_i$ $(|i-j| > 1)$,
  $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ $(i \neq 0)$
  $T_0T_1T_0T_1 = T_1T_0T_1T_0$
- X_i's commute each other,
- $T_iX_iT_i = X_{i+1}$ $(i \neq 0), T_0X_1^{-1}T_0 = X_1$,
  $T_iX_j = X_jT_i$ $(j \neq i, i+1)$

K_n^B := the Grothendieck group of the abelian category of finite-dimensional H_n^B-modules

It has a basis $\Big\{[S]; [S]$ is a simple H_n^B-modules$\Big\}$

$\mathsf{K}^B := \bigoplus_n \mathsf{K}_n^B$ has a structure of a (right) K^A-module.

H_{m+n}^B is generated by

$$\begin{array}{l} T_0, T_1, \ldots, T_{m-1},\ T_m,\ T_{m+1}, \ldots, T_{m+n-1} \\ \qquad \underbrace{X_1, X_2, \ldots, X_m,}_{\mathsf{H}_m^B}\ \underbrace{X_{m+1}, \ldots\ldots\ldots, X_{m+n}}_{\mathsf{H}_n^A} \end{array}$$

$\mathsf{H}_m^B \otimes \mathsf{H}_n^A \subset \mathsf{H}_{m+n}^B$

For an H_m^B-module M and an H_n^A-module N

$$\begin{array}{ccccc} [M] & \cdot & [N] & = & \Big[\mathrm{Ind}_{\mathsf{H}_m^B \otimes \mathsf{H}_n^A}^{\mathsf{H}_{m+n}^B}(M \otimes N)\Big] \\ \Cap & & \Cap & & \Cap \\ \mathsf{K}_m^B & & \mathsf{K}_n^A & & \mathsf{K}_{m+n}^B \end{array}$$

Symmetric crystal

θ: a Dynkin diagram involution $I \to I$
without fixed points

$E_i, F_i, K_i^{\pm 1} \in \mathrm{End}_K(U_q^-(\mathfrak{g}))$ defined by

$$F_i(a) = f_i a + (\mathrm{Ad}(t_i)a) f_{\theta(i)}, \quad E_i(a) = e_i' a$$

$$K_i(a) := q^{(\alpha_i + \alpha_{\theta(i)}, \alpha)} a \quad \text{for } a \in U_q^-(\mathfrak{g})_\alpha$$

$$E_i \circ F_j = q^{-(\alpha_i, \alpha_j)} F_j \circ E_i + \delta_{i,j} + \delta_{\theta(i),j} K_i$$

$K_i = K_{\theta(i)}$:

$$\begin{aligned} K_i E_j K_i^{-1} &= q^{(\alpha_i, \alpha_j + \alpha_{\theta(j)})} E_j, \\ K_i F_j K_i^{-1} &= q^{-(\alpha_i, \alpha_j + \alpha_{\theta(j)})} F_j, \end{aligned}$$

$\mathcal{B}_\theta(\mathfrak{g})$: the subalgebra of $\mathrm{End}(U_q^-(\mathfrak{g}))$
generated by E_i, F_i, K_i

V_θ: the sub-K-module of $U_q^-(\mathfrak{g})$ generated by 1 under the actions of the F_i's,

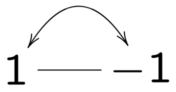
q-boson $\quad E_iF_i = q_i^{-2}F_iE_i + 1$

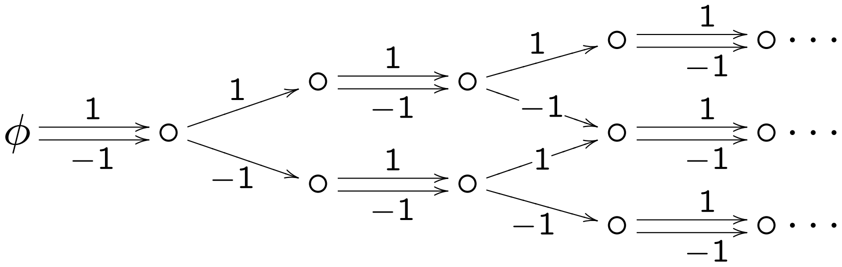
V_θ is stable by E_i's

We define $\tilde{E}_i$, $\tilde{F}_i$ similarly to $\tilde{e}_i$ and $\tilde{f}_i$

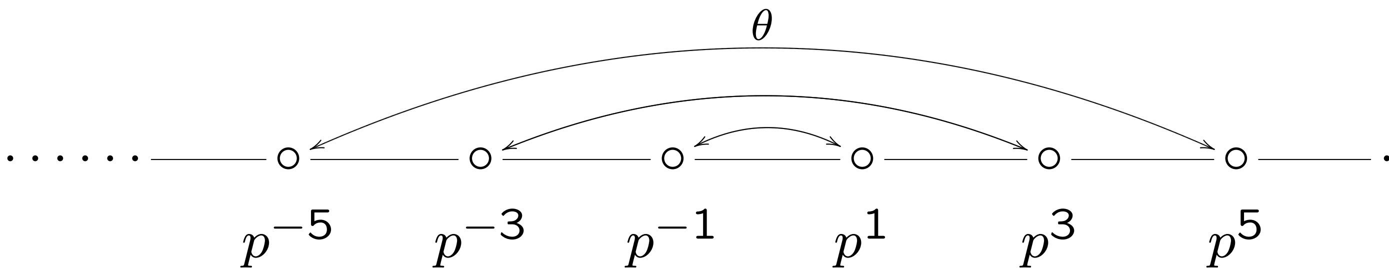
Conj. V_θ has a crystal basis and a global basis.

Th. (Enomoto) the conjecture holds in the symmetric Cartan matrix case.





$I = \mathbb{C}^*$ as type A-case, $\theta(a) = a^{-1}$



Conj.(rough statement) math.RT/0608079

(i) $\mathbb{C}\otimes_{\mathbb{Z}}\mathsf{K}^{B}$ is isomorphic to the specialization of V_θ at $q = 1$,

(ii) the irreducible modules correspond to the upper global basis (at $q{=}1$).