Odd categorification of quantum groups

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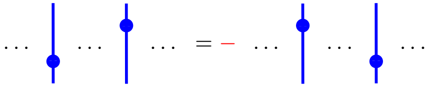
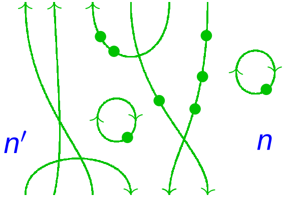


April 21st, 2012

First let's look at some applications of categorified quantum groups.



$$\mathcal{E}^3\mathcal{F}^3 1_n\{s'\}$$
$$\Uparrow$$
$$\mathcal{E}^3\mathcal{F}^3 1_n\{s\}$$

Braid Group actions

- The Weyl group action on $\mathfrak{g}$ gives rise to an isomorphism between certain weight spaces in a $\mathbf{U}(\mathfrak{g})$-module.
- When we pass from $\mathbf{U}(\mathfrak{g})$ to $\mathbf{U}_q(\mathfrak{g})$ the Weyl group action becomes a braid group action.

$\mathfrak{g} = \mathfrak{sl}_2$

The Weyl group $W = S_2$ gives isomorphisms

$$\begin{tikzcd}
\cdots & V_{-n+2} \arrow[r, bend left, "E"] & V_{-n} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[rrrr, leftrightarrow, bend left=40, "\tau"] & V_{-n-2} \arrow[l, bend left, "F"] & \cdots & V_{n-2} \arrow[r, bend left, "E"] & V_n \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V_{n+2} \arrow[l, bend left, "F"] & \cdots
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We can express the action of the Weyl group as

$$\tau 1_n = \sum_{s \geq 0} (-q)^s F^{(n+s)} E^{(s)} 1_n .$$

Categorifying the reflection element

We can lift this element to a *complex*

$$\tau \mathbb{1}_n := \mathcal{F}^{(n)} \longrightarrow \mathcal{F}^{(n+1)}\mathcal{E} \longrightarrow \dots \longrightarrow \mathcal{F}^{(n+s)}\mathcal{E}^{(s)} \longrightarrow \dots$$

The differential is defined using the 2-morphisms $\mathcal{U}(\mathfrak{sl}_2)$.

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Chuang-Rouquier/Cautis-Kamnitzer-Licata

Given an action of the 2-category $\mathcal{U}(\mathfrak{g})$ on an additive category $\mathcal{V}$ the functor of tensoring with the complex $\tau\mathbb{1}_n$ gives rise to derived equivalences

$$\begin{tikzcd}
\cdots & D(\mathcal{V}_{-n+2}) \arrow[r, bend left, "\mathcal{E}"] & D(\mathcal{V}_{-n}) \arrow[l, bend left, "\mathcal{F}"] \arrow[r, bend left, "\mathcal{E}"] \arrow[rrr, leftrightarrow, bend left=40] & D(\mathcal{V}_{-n-2}) \arrow[l, bend left, "\mathcal{F}"] \cdots & D(\mathcal{V}_{n-2}) \arrow[r, bend left, "\mathcal{E}"] & D(\mathcal{V}_{n}) \arrow[l, bend left, "\mathcal{F}"] \arrow[r, bend left, "\mathcal{E}"] & D(\mathcal{V}_{n+2}) \arrow[l, bend left, "\mathcal{F}"] & \cdots
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Derived equivalences

- Chuang and Rouquier showed that derived equivalences could be constructed in the context of abelian categories.
- Cautis, Kamnitzer and Licata showed that the higher structure of $\mathcal{U}(\mathfrak{sl}_n)$ gives derived equivalences in the more general setting of triangulated categories.

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- Used by CR to prove the Abelian defect conjecture for symmetric groups.
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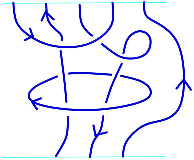
Applications

- Used by CR to prove the Abelian defect conjecture for symmetric groups.
- Used by CKL to construct derived equivalences between derived categories of coherent sheaves on cotangent bundles to Grassmannians.
- Cautis-Kamnitzer's geometric construction of Khovanov homology and related invariants for $\mathfrak{sl}_n$ can be understood as arising from these derived equivalences via a categorified version of skew-Howe duality.

Knot theory and Categorification

Jones polynomial



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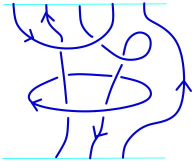

Representation theory of quantum $\mathfrak{sl}_2$

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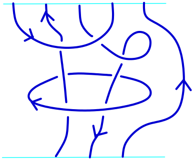

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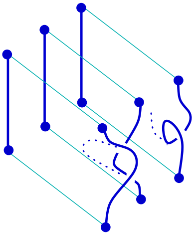

Khovanov homology

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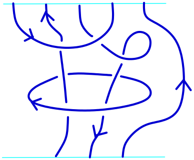

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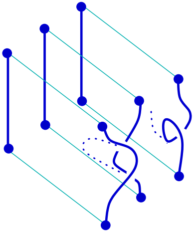

Categorified representation theory

Khovanov homology

Motivation from Knot theory

$$\begin{array}{ccc}
\text{Jones polynomial} & \longleftrightarrow & \text{Rep theory of quantum } \mathfrak{sl}_2 \\
\downarrow & & \downarrow \\
\text{Khovanov homology} & \longleftrightarrow & \text{Categorified Rep theory of } \mathfrak{sl}_2
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The discovery of Khovanov homology motivated the study of categorified quantum $\mathfrak{sl}_2$.

Motivation from Knot theory

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The discovery of Khovanov homology motivated the study of categorified quantum $\mathfrak{sl}_2$.

This categorification is closely connected to

- The geometry of flag varieties and Grassmannians
- The combinatorics of symmetric functions
- Hecke algebras in type A

Odd Khovanov homology

Khovanov's categorification of the Jones polynomial is not unique.

Ozsváth, Rasmussen, Szabó

$$\begin{tikzcd}
\text{Khovanov homology} \arrow[dr] \arrow[rr, phantom, "\neq"] & & \text{Odd Khovanov homology} \arrow[dl] \\
& \text{Jones polynomial} &
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Decategorification

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- Both theories categorify the Jones polynomial

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- Both theories categorify the Jones polynomial
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\text{Khovanov homology} \arrow[dr, color=red] & \neq & \text{Odd Khovanov homology} \arrow[dl, color=red] \\
& \text{Jones polynomial} &
\end{tikzcd}$$

Decategorification

- Both theories categorify the Jones polynomial
- Both theories agree when coefficients are reduced modulo two
- Shumakovitch showed that both theories are distinct

Idea: Utilize these discoveries in knot theory to discover new structures in geometric representation theory via the connection to quantum groups

$$\begin{tikzcd}
 & \text{Jones polynomial} \arrow[rr, leftrightarrow] \arrow[d] \arrow[ddl] & & \text{Rep theory of quantum } \mathfrak{sl}_2 \arrow[d] \arrow[ddl] \\
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This suggests a program of identifying “odd” analogs of categorified quantum groups and related objects in geometric representation theory.

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- These theories should be distinct from classical theories.
- They should agree with the classical theories when coefficients are reduced mod 2.
- Odd theories should have many of the same combinatorial features as their classical counterparts.
- Noncommutativity will be an inherent feature of such oddifications.

Quantum $\mathfrak{sl}_2$

There is a triangular decomposition

$$\mathbf{U}_q(\mathfrak{sl}_2) = \mathbf{U}_q^+(\mathfrak{sl}_2) \otimes \mathbf{U}_q(\mathfrak{h}) \otimes \mathbf{U}_q^-(\mathfrak{sl}_2)$$

For categorification it is best to work integrally $\mathbb{Z}[q, q^{-1}]$.

$$\mathbf{U}_q^+(\mathfrak{sl}_2)_{\mathbb{Z}} = \left\langle E^{(n)} \mid n \in \mathbb{N} \right\rangle$$

where $E^{(n)}$ is the divided power

$$E^{(n)} = \frac{E^n}{[n]!}.$$

$$
\mathbf{U}_q^+(\mathfrak{sl}_2)_{\mathbb{Z}} \quad \underset{\text{Grothendieck ring}}{\overset{\text{Categorification}}{\rightleftarrows}} \quad \mathcal{U}_q^+(\mathfrak{sl}_2)_{\mathbb{Z}}
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\begin{array}{rcl}
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$$E^n \mapsto \mathcal{E}^n$$

$$E^{(n)} \mapsto \mathcal{E}^{(n)}$$

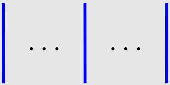
We must also categorify the single relation

$$E^{(n)} = \frac{E^n}{[n]!}, \qquad \text{or} \qquad E^n = [n]!E^{(n)}$$

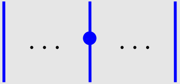
where addition becomes direct sums of objects and equalities become explicit isomorphisms

$$\mathcal{E}^n \cong \bigoplus_{n!} \mathcal{E}^{(n)}$$

Generators for the NilHecke algebra



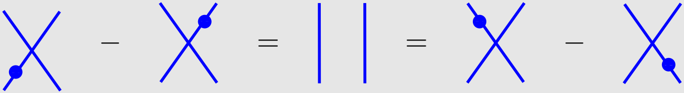
$$:= \quad 1 \in \mathcal{NH}_a$$



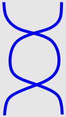
$$:= \quad x_r$$

$$:= \quad \partial_r$$

Relations

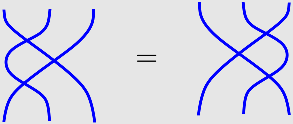


$$\partial_r x_r - x_{r+1}\partial_r = 1 = x_r\partial_r - \partial_r x_{r+1}$$



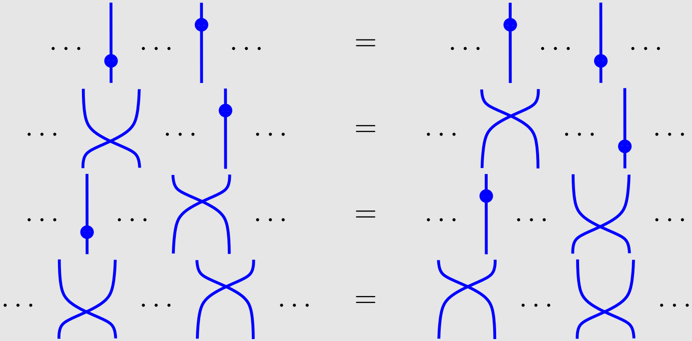
$$= \quad 0$$

$$\partial_r\partial_r = 0$$



$$\partial_r\partial_{r+1}\partial_r = \partial_{r+1}\partial_r\partial_{r+1}$$

Isotopy relations



Algebraic Isotopy Relations

$$x_i x_j = x_j x_i \quad (i \neq j),$$
$$\partial_i \partial_j = \partial_j \partial_i \quad (|i-j| > 1),$$
$$x_i \partial_j = \partial_j x_i \quad (i \neq j, j+1).$$

Polynomial representation

The algebra $\mathcal{NH}_n$ acts on the polynomial ring $\mathrm{Pol}_n := \mathbb{Z}[x_1, x_2, \dots, x_n]$ with x_i acting by multiplication and ∂_i acting by divided difference operators

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} \qquad f \in \mathrm{Pol}_n,$$

$s_i(f)$ is the action of the symmetric group S_n by permuting variables.

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Alternatively, we can define ∂_i by

$$\partial_i(1) = 0, \qquad \partial_i(x_j) = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \text{ for all } f, g \in \mathbb{Z}[x_1, \dots, x_a].$$

Symmetric functions

The ring of symmetric functions has many descriptions

$$\Lambda_n = \mathbb{Z}[x_1, x_2, \dots, x_n]^{S_n} = \bigcap_{i=1}^{n-1} \ker(\partial_i) = \bigcap_{i=1}^{n-1} \mathrm{im}\,(\partial_i).$$

This ring can also be described as

$$\Lambda_n \cong \mathbb{Z}[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n],$$

where ε_k is the usual elementary symmetric polynomial

$$\varepsilon_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_n}$$

of degree $2k$ (since $\deg(x_i) = 2$).

Example ($n = 3$)

$$\begin{aligned}
\varepsilon_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\
\varepsilon_2(x_1, x_2, x_3) &= x_1x_2 + x_2x_3 + x_1x_3 \\
\varepsilon_1(x_1, x_2, x_3) &= x_1x_2x_3
\end{aligned}$$

There are other natural bases for Λ_n such as

- complete symmetric functions
- Schur functions

$$s_\lambda = \partial_{w_0}(x_1^{n-1+\lambda_1} x_2^{n-2+\lambda_2} \dots x_n^{\lambda_n})$$

where w_0 is the longest element of the symmetric group.

The ring of polynomials Pol_n is a free Λ_n-module of rank $n!$. Two natural basis for Pol_n as a free Λ_n module are

- The set $\left\{x_1^{\ell_1}x_2^{\ell_2}\dots x_n^{\ell_n}\right\}$ where $0 \leq \ell_i \leq n-i$.
- The basis of Schubert polynomials

$$\mathfrak{S}_w := \partial_{w_0 w^{-1}}(x_1^{n-1}x_2^{n-2}\dots x_n^0)$$

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Example

$x_n = \varepsilon_1 \cdot 1 - 1 \cdot x_1 - 1 \cdot x_2 - \cdots - 1 \cdot x_{n-1}$

We can think of a polynomial $f \in \mathrm{Pol}_n$ as an $n!$-dimensional vector with coefficients in the ring Λ_n. I.e. $x_n = \begin{pmatrix} \varepsilon_1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$.

A Λ_n-module endomorphism of Pol_n is just an $n! \times n!$ matrix with coefficients in Λ_n.

The action of $\mathcal{NH}_n$ on Pol_n gives rise to a homomorphism

$$\mathcal{NH}_n \longrightarrow \mathrm{End}_{\Lambda_n}(\mathrm{Pol}_n) \cong \mathrm{Mat}(n!, \Lambda_n)$$

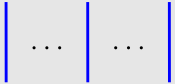
One can show that this map is an isomorphism.

Theorem

There is an isomorphism

$$\begin{array}{rcl} \bigoplus_{n\in\mathbb{N}} K_0\left(\mathcal{NH}_n - \mathrm{pmod}\right) & \longrightarrow & \mathbf{U}^+(\mathfrak{sl}_2) \\ \mathcal{NH}_n & \mapsto & E^n \\ \mathcal{NH}_n e_{1,1} & \mapsto & E^{(n)} \end{array}$$

Odd NilHecke Generators



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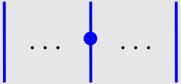

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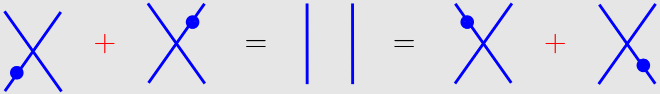

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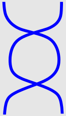
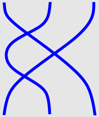
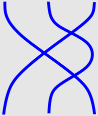


$$= \quad 0$$

$$\partial_r \partial_r = 0$$



$$=$$



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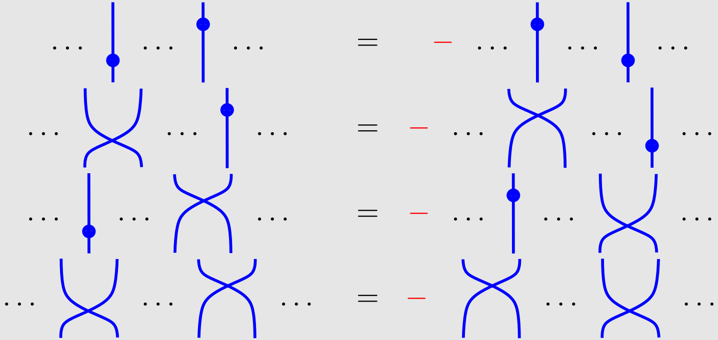
Isotopy relations



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Skew Polynomial representation

Define the ring of *odd polynomials* to be

$$\mathrm{OPol}_a = \mathbb{Z}\langle x_1, \dots, x_a \rangle / \langle x_i x_j + x_j x_i = 0 \text{ for } i \neq j \rangle.$$

The symmetric group S_a acts as the ring endomorphism

$$s_i(x_j) = \begin{cases} -x_{i+1} & \text{if } j = i, \\ -x_i & \text{if } j = i+1, \\ -x_j & \text{otherwise.} \end{cases}$$

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The *odd divided difference operators* are the linear operators ∂_i defined by

$$\partial_i(1) = 0,$$
$$\partial_i(x_j) = \begin{cases} 1 & \text{if } j = i, i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \text{ for all } f, g \in \mathbb{Z}\langle x_1, \dots, x_a \rangle.$$

Odd Symmetric functions

Define the ring of *odd symmetric polynomials* to be the subring

$$\mathrm{O}\Lambda_n = \bigcap_{i=1}^{n-1} \ker(\partial_i) = \bigcap_{i=1}^{n-1} \mathrm{im}\,(\partial_i)$$

of OPol_n.

By analogy with the even case, we introduce the *odd elementary symmetric polynomials*

$$\varepsilon_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k}, \qquad \textit{where } \widetilde{x}_i = (-1)^{i-1} x_i$$

Example (n=3)

$$\begin{aligned} \varepsilon_1 &= x_1 - x_2 + x_3 \\ \varepsilon_2 &= -x_1x_2 + x_2x_3 - x_2x_3 \\ \varepsilon_3 &= -x_1x_2x_3 \end{aligned}$$

Proposition (Ellis, Khovanov, L)

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- Products $\varepsilon_\lambda = \varepsilon_{\lambda_1}\varepsilon_{\lambda_2}\dots\varepsilon_{\lambda_n}$ for partitions λ form a basis for $\mathrm{O}\Lambda_n$. There are other basis corresponding to complete and Schur symmetric functions with closely related combinatorics.
- The rings Λ_n and $\mathrm{O}\Lambda_n$ have the same graded rank and become isomorphic when coefficients are reduced modulo two.

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Remark

Note that odd symmetric functions are not the invariants (or antinvariants) for an action of S_n.

Proposition

The ring of odd polynomials OPol_n is a free left (resp. right $\mathrm{O}\Lambda_n$) module with basis given by odd Schubert polynomials

$$\mathfrak{S}_w := \partial_{w_0 w^{-1}}(x_1^{n-1} x_2^{n-2} \dots x_n^0)$$

This allows us to identify the endomorphism ring $\mathrm{End}_{\mathrm{O}\Lambda_n}(\mathrm{OPol}_n)$ as a matrix ring $Mat(n!, \mathrm{O}\Lambda_n)$. The action of $\mathcal{ONH}_n$ on odd polynomials gives rise to

Theorem (Ellis,Khovanov, L)

There is an isomorphism

$$\begin{array}{rcl} \bigoplus_{n \in \mathbb{N}} K_0\left(\mathcal{ONH}_n - \mathrm{pmod}\right) & \longrightarrow & \mathbf{U}^+(\mathfrak{sl}_2) \\ \mathcal{ONH}_n & \mapsto & E^n \\ \mathcal{ONH}_n e_{1,1} & \mapsto & E^{(n)} \end{array}$$

Covering Kac-Moody algebras

The existence of the even and the odd theories has a representation theoretic explanation via the work of Hill-Wang and Clark-Wang.

Introduce a parameter π with $\pi^2 = 1$.

Covering Kac-Moody algebras

$$\begin{tikzcd}
 & \mathbf{U}_q^{\pi} \arrow[dl, "\pi \to 1"'] \arrow[dr, "\pi \to -1"] & \\
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- There is a novel new bar involution $\overline{q} = \pi q^{-1}$.
- This leads to the first construction of canonical bases for super Lie algebras! (Positive parts for super Lie algebras Hill-Wang, entire quantum group in rank 1 by Clark-Wang.)

Cyclotomic quotients (even case)

Given an integer $N \in \mathbb{N}$ we can define the cyclotomic quotient $\mathcal{NH}_n^N$ by quotienting by the ideal $\langle x_1^N \rangle$.

Theorem

There is an isomorphism

$$\bigoplus_{n \in \mathbb{N}} K_0\left(\mathcal{NH}_n^N - \mathrm{pmod}\right) \quad \longrightarrow \quad V_N$$

where V_N is the integral version of the irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-module of highest weight N.

This result relies on the fact that $\mathcal{NH}_n^N$ is Morita equivalent to the cohomology ring of the Grassmannian $Gr(k; N)$ of k-planes in $\mathbb{C}^N$.

Odd cyclotomic quotients

Odd cyclotomic quotients $\mathcal{ONH}_n^N$ can be defined in the same way as ordinary cyclotomic quotients.

- Odd cyclotomic quotients also categorify irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-representations.

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- The ring $OH^*(Gr(k;N))$ has the same graded rank as $H^*(Gr(k;N))$ and these rings become isomorphic when coefficients are reduced modulo two.
- The ring $OH^*(Gr(k;N))$ has a basis of appropriate odd Schur functions.

Other odd cohomologies

The full flag variety X consists of the set of all flags in $\mathbb{C}^n$,

$$X = \left\{0 = U_0 \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n \mid \dim_{\mathbb{C}} U_i = i\right\}.$$

The Springer variety X^λ is the closed subvariety of X consisting of those flags preserved by a nilpotent matrix x^λ of Jordan type λ

$$X^\lambda = \left\{(U_0, U_1, \dots, U_n) \in X \mid x^\lambda U_i \subseteq U_{i-1}\right\}.$$

The cohomology Springer varieties carry an action of the symmetric group S_n.

$$H^{top}(X^\lambda) \cong S_\lambda$$

where S_λ is the irreducible S_n-module corresponding to λ.

Khovanov homology and Springer theory

Springer varieties have a close connection to Khovanov homology suggesting they should also have natural odd analogs.

1. Khovanov showed that the center of rings H_n is isomorphic to the cohomology of the (n, n)-Springer variety.
2. Using convolution algebras Stroppel and Webster relate the entire cohomology of the (n, n)-Springer variety to a version of Khovanov's arc algebra.
3. The geometric construction of Khovanov homology by Cautis and Kamnitzer utilizes Springer varieties as well.

Theorem (L,Russell)

There exist oddifications $OH(X^\lambda)$ of the cohomology of Springer varieties.

Theorem (L,Russell)

There is an action of the Hecke algebra $H_{-1}(n)$ at $q=-1$ on the odd cohomology of the Springer variety $OH(X^{\lambda})$. The top degree cohomology is isomorphic to the corresponding Specht module of the Hecke algebra.

The odd symmetric group is the Hecke algebra at $q=-1$

Theorem (L,Russell)

The ring of odd symmetric functions are precisely the invariants for an action of $H_{-1}(n)$ on the odd polynomial ring.