$A = (a_{ij})_{n \times n}$ symmetrizable GCM

$\mathfrak{g} = \mathfrak{g}(A) = \langle \mathfrak{h}, e_i, f_i \mid 1 \le i \le n \rangle$

Kac-Moody Lie alg. associated with A.

$\mathfrak{g}^+ = \langle e_i \mid 1 \le i \le n \rangle \subset \mathfrak{g}$

$\mathfrak{g}^- = \langle f_i \mid 1 \le i \le n \rangle \subset \mathfrak{g}$

$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$

Recall $\dim \mathfrak{h} = n + \text{corank}(A)$

$h_i = [e_i, f_i], \quad 1 \le i \le n$

$h_i \in \mathfrak{h}, \quad 1 \le i \le n$

$\alpha \in \Delta \subset \mathfrak{h}^*$ set of roots

$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h) x \ \forall h \in \mathfrak{h} \}$

$\mathfrak{g}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$

$\mathfrak{g}^\pm$ are nilpotent subalg. of $\mathfrak{g}$.

$\mathfrak{h} \oplus \mathfrak{g}^+$ is a solvable subalg. of $\mathfrak{g}$ called the Borel subalg.

$\mathcal{U}(\mathfrak{g})$ universal enveloping alg. which is an assoc. alg. with unity and $j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ Lie alg. hom.

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I$$

$$I = \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g} \rangle$$

Fix ordered basis $\overset{\mathcal{B}}{\shortparallel}\{x_i \mid i \in I\}$ for $\mathfrak{g}$.

Write $j(x_i) \in \mathcal{U}(\mathfrak{g})$ as x_i and suppress the $\otimes$, i.e

$$y_1 \otimes y_2 \otimes \cdots \otimes y_k = y_1 y_2 \cdots y_k \;\forall\, y_i \in \mathfrak{g}.$$

For $y+I \in \mathcal{U}(\mathfrak{g})$, write $y+I$ as y.

Then $\mathcal{U}(\mathfrak{g}) = \text{span}\{x_{i_1} x_{i_2} \cdots x_{i_k} \mid k \geq 0, x_{i_j} \in \mathcal{B}\}$

<u>Thm</u>: $\mathcal{U}(\mathfrak{g})$ has basis $\{x_{i_1} x_{i_2} \cdots x_{i_k} \mid k \geq 0, i_1 \leq i_2 \leq \cdots \leq i_k, x_{i_j} \in \mathcal{B}\}$

<u>Cor</u>: (PBW) $j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is 1-1.

• $\cancel{y_1 y_2 \cdots y_k}$

Length of a monomial

$$\ell(x_{i_1} x_{i_2} \cdots x_{i_k}) = k \quad , \quad x_{i_j} \in \mathcal{B}.$$

$inv(x_{i_1} x_{i_2} \cdots x_{k})$ = # of inversions in the sequence $\{i_1, i_2, \cdots, i_k\}$

$$= \left|\{(a,b) \mid 1 \le a \le b \le k, \ i_a > i_b\}\right|$$

Pf of Thm (Spanning)

Suppose $x_{i_1} x_{i_2} \cdots x_{i_k}$ is not ordered (i.e. it is not a standard monomial)

Assume $x_{i_1} x_{i_2} \cdots x_{i_k}$ is a nonstandard monomial with minimum length and minimum number of inversions.

Since $x_{i_1} \cdots x_{i_k}$ is nonstandard so $\exists$ 'a' such that $i_a > i_{a+1}$.

Then

$$x_{i_1} x_{i_2} \cdots (x_{i_a} x_{i_{a+1}}) \cdots x_{i_k} = x_{i_1} \cdots (x_{a+1} x_{i_a} + [x_{i_a}, x_{i_{a+1}}]) \cdots x_{i_k}$$

$$= \underbrace{x_{i_1} \cdots x_{i_{a-1}} x_{i_{a+1}} x_{i_a} \cdots x_{i_k}}_{\text{has less \# of inversions}} + \underbrace{x_{i_1} \cdots x_{i_{a-1}} [x_{i_a}, x_{i_{a+1}}] \cdots x_{i_k}}_{\text{lin. comb. of mono. of shorter length}}$$

By induction on length and # of inversions the RHS is a lin. comb. of standard monomials.

Hence $\mathcal{U}(\mathfrak{g}) = \text{span}\{x_{i_1} \cdots x_{i_k} \mid k \geq 0, i_1 \leq \cdots \leq i_k, x_{i_j} \in \mathcal{B}\}$

Left to show that the standard monomials are linearly indep.

(Linear Independence):

Let V be a vector space $/F$ with basis $\{z_M \mid M = (i_1, i_2, \ldots, i_k), k \geq 0, i_j \in I, i_1 \leq i_2, \ldots, i_k\}$

Step 1 Make V a $\mathfrak{g}$-module.

For $i \in I$, define $i \leq \phi$ $\forall i$

and $i \leq M = (i_1, i_2, \ldots, i_k) \neq \phi$ if $i \leq i_1$

Define $x_i \cdot z_M$ inductively as follows:

- $x_i \cdot z_\phi = z_i = z_{(i)}$

Assume that:

(1) $x_j \cdot z_N$ has been defined $\forall j \in I$ when $l(N) < l(M)$ and $\forall j < i$ when $l(N) = l(M)$.

(2) $l_j \cdot z_N$ is a linear combination of vectors of the form z_R where $l(R) \leq l(N)+1$.

define

$$x_i \cdot z_M = \begin{cases} z_i z_M = z_{(i,M)} = z_i z_{i_1} \cdots z_{i_k} & \text{if } i \leq M \\ x_{i_1} \cdot (x_i \cdot z_{i_2} \cdots z_{i_k}) + [x_i, x_{i_1}] \cdot z_{i_2} \cdots z_{i_k} & \text{if } i > M. \end{cases}$$

Lemma (hard) Under the above action V is a $\mathfrak{g}$-module.

Pf (skip)

Denote the standard monomial

$x_{i_1} x_{i_2} \cdots x_{i_k}$ by x_M, $M = (i_1, i_2, \cdots, i_k)$

and $x_\phi = 1$

Claim: $x_M \cdot z_\phi = z_M$ for all $M = (i_1, i_2, \cdots, i_k)$

Use induction on $l(M)$.

If $l(M) = 0 \Rightarrow M = \phi$ and

$$x_\phi \cdot z_\phi = 1 \cdot z_\phi = z_\phi$$

If $l(M) = 1 \Rightarrow M = (i)$ and

$$x_i \cdot z_\phi = z_{(i)}$$

Assume $l(M) > 1$. Write

$$x_M = x_{i_1} x_N \quad , \quad N = (i_2, i_3, \cdots, i_k)$$

Then $x_M \cdot z_\phi = x_{i_1} \cdot (x_N \cdot z_\phi) = x_{i_1} \cdot z_N$ by induction

$$= z_M \quad \text{since } i_1 \leq N$$

$\Rightarrow$ Claim.

Suppose $\sum_M c_M x_M = 0$

$$\Rightarrow \left(\sum_M c_M x_M \right) \cdot z_\phi = 0$$

$$\Rightarrow \sum_M c_M (x_M \cdot z_\phi) = 0$$

$$\Rightarrow \sum_M c_M z_M = 0 \Rightarrow c_M = 0 \ \forall M$$

$\Rightarrow \{ x_{i_1} \cdots x_{i_k} \mid k \geq 0, \ i_1 \leq \cdots \leq i_k, \ i_j \in I \}$ lin. indep. hence a basis for $\mathcal{U}(\mathfrak{g})$. //

Cor: $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$

$\Longrightarrow \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_1)\mathcal{U}(\mathfrak{g}_2) \cdots \mathcal{U}(\mathfrak{g}_m)$

In particular,

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$$

$\Longrightarrow \quad \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}^-)\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{g}^+)$

Cor: V is $\mathfrak{g}$-module $\Longleftrightarrow$ V is a $\mathcal{U}(\mathfrak{g})$-module.

Pf: V $\mathfrak{g}$-module

$\Longrightarrow \exists$ repn. $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\phi"] \arrow[dr, dashed, "j"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathfrak{gl}(V) \\ & \mathcal{U}(\mathfrak{g}) \arrow[ur, dashed, "\exists!\,\psi"'] & \end{tikzcd}$$

$\exists!\,\psi$ assoc. alg. hom

i.e. $\quad \psi j = \phi$

We $\psi : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$ assoc. alg. hom., hence V is a $\mathcal{U}(\mathfrak{g})$-module.