$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$A_1^{(1)} = \mathfrak{g}(A) = \mathfrak{sl}(2,\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$\Delta^+ = \{\alpha_0 + k\delta, \alpha_1 + k\delta \mid k \geq 0\} \cup \{k\delta \mid k > 0\}$$

$$\Delta = \Delta^+ \cup -\Delta^+$$

$$W = \langle r_0, r_1 \rangle = \{(r_0 r_1)^m, r_0 (r_0 r_1)^m \mid m \in \mathbb{Z}\}$$

Denominator formula:

$$\prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{\dim \mathfrak{g}_\alpha} = \sum_{w \in W} (-1)^{\ell(w)} e(w\rho - \rho)$$

where $\rho \in \mathfrak{h}^*$ s.t. $\rho(h_i) = 1$, $i = 0, 1$.

In This case $\dim \mathfrak{g}_\alpha = 1$ $\forall \alpha \in \Delta^+$

Set $u = e(-\alpha_0)$, $v = e(-\alpha_1)$

$$\Longrightarrow \quad e(-\delta) = e(-\alpha_0 - \alpha_1) = e(-\alpha_0)e(-\alpha_1) = uv.$$

$$e(-\alpha_0 - k\delta) = e(-\alpha_0)\big(e(-\delta)\big)^k = u(uv)^k$$
$$= u^{k+1} v^k$$

$$e(-\alpha_1 - k\delta) = v(uv)^k = u^k v^{k+1}$$

$$e(-k\delta) = (uv)^k$$

$$LHS = \prod_{k\geq 0} (1-u^{k+1}v^k)(1-u^k v^{k+1}) \prod_{k>0} (1-u^k v^k)$$

$$= \prod_{k\geq 1} (1-u^k v^{k-1})(1-u^{k-1}v^k)(1-u^k v^k)$$

$$RHS = \sum_{m\in\mathbb{Z}} (-1)^m u^{m(m+1)/2} v^{m(m-1)/2}$$

Which gives the Jacobi Triple Product identity:

$$\prod_{k\geq 1} (1-u^k v^{k-1})(1-u^{k-1}v^k)(1-u^k v^k)$$
$$= \sum_{m\in\mathbb{Z}} (-1)^m u^{\binom{m+1}{2}} v^{\binom{m}{2}} .$$

$w = 1$

$e(\rho - \rho) = e(0) = 1$

$w = r_0$

$e(r_0\rho - \rho) = e(\rho - \rho(h_0)\alpha_0 - \rho) = e(-\alpha_0) = u$

$w = r_0 r_1$

$e(r_0 r_1 \rho - \rho) = e(r_0(\rho - \rho(h_1)\alpha_1) - \rho)$

$= e((\rho - \alpha_1) - (\rho - \alpha_1)(h_0)\alpha_0 - \rho)$

$= e(\rho - \alpha_1 - 3\alpha_0 - \rho) = e(-3\alpha_0 - \alpha_1) = u^3 v$

$w = (r_0 r_1)^2$

$e((r_0 r_1)^2 \rho - \rho) = e(r_0 r_1(\rho - \alpha_1 - 3\alpha_0) - \rho)$

$= e(r_0((\rho - \alpha_1 - 3\alpha_0) - (\rho - \alpha_1 - 3\alpha_0)(h_1)\alpha_1) - \rho)$

$= e(r_0(\rho - 6\alpha_1 - 3\alpha_0) - \rho)$

$= e((\cancel{\rho} - 6\alpha_1 - 3\alpha_0) - (\rho - 6\alpha_1 - 3\alpha_0)(h_0)\alpha_0 - \cancel{\rho})$

$= e(-6\alpha_1 - 10\alpha_0) = u^{10} v^6$

Let $A = (a_{ij})_{n\times n}$ symmetrizable, indecomposable GCM.

Let $\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i \mid 1 \le i \le n \rangle$

where $\dim \mathfrak{h} = n + \text{corank}(A)$

satisfying the 6 Serre relations

$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ simple roots

$\check{\Pi} = \{h_1, h_2, \cdots, h_n\}$ simple coroots.

$\alpha_j \in \mathfrak{h}^*$, $\alpha_j(h_i) = a_{ij}$ $1 \le i, j \le n$

$\mathfrak{g}_{\alpha_j} = \text{span}\{e_j\}$, $\mathfrak{g}_{-\alpha_j} = \text{span}\{f_j\}$

$h_j = [e_j, f_j]$, $1 \le j \le n$.

$\mathfrak{g}$ is known as the Kac-Moody Lie algebra associated with $A = (a_{ij})_{n\times n}$.

Universal Enveloping algebra $\mathcal{U}(\mathfrak{g})$ associated with the Lie alg. $\mathfrak{g}$:

<u>Defn</u>: It is a pair $(\mathcal{U}(\mathfrak{g}), j)$ such that $\mathcal{U}(\mathfrak{g})$ is an associative algebra, $j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is a Lie alg. homomorphism satisfying the following universal property:

If (A, φ) is any pair where A is an assoc. alg. and $\varphi : \mathfrak{g} \longrightarrow A$ is a Lie algebra homomorphism,

$$\begin{tikzcd} & A \\ \mathfrak{g} \arrow[ur, "\varphi"] \arrow[r, "j"'] \arrow[r, phantom, shift left=3ex, "\circlearrowleft"] & \mathcal{U}(\mathfrak{g}) \arrow[u, dashed, "\exists!\,\psi,\ \text{assoc. alg. hom}"'] \end{tikzcd}$$

then there exists unique assoc. alg. hom $\psi : \mathcal{U}(\mathfrak{g}) \to A$ such that

$$\psi j = \varphi .$$

Existence of Univ. enveloping algebra of the Lie alg. $\mathfrak{g}$.

Consider the Tensor algebra

$$(T(\mathfrak{g}), i)$$

Recall $i : \mathfrak{g} \longrightarrow T(\mathfrak{g})$ linear map

$$T(\mathfrak{g}) = \bigoplus_{n \geq 0} T^n(\mathfrak{g})$$

$$T^0(\mathfrak{g}) = F \,, \; T^1(\mathfrak{g}) = \mathfrak{g}$$

$$T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n \text{ times}}$$

$$T^n(\mathfrak{g}) \cdot T^m(\mathfrak{g}) \subseteq T^{m+n}(\mathfrak{g})$$

For $x, y \in \mathfrak{g}$

$$i(x), i(y) \in T^1(\mathfrak{g})$$

$$\underbrace{[i(x), i(y)]}_{\cap \; T^2(\mathfrak{g})} \overset{?}{\neq} i([x,y]) \in T^1(\mathfrak{g})$$

i.e. $i : \mathfrak{g} \to T(\mathfrak{g})$ not a Lie alg. hom.

Consider the ideal I of $T(\mathfrak{g})$ defined by

$$I = \text{gen. by } \{ i(x) \otimes i(y) - i(y) \otimes i(x) - i([x,y]) \mid x, y \in \mathfrak{g} \}$$

Define

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / I$$

$j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ given by

$$j(x) = i(x) + I \quad \forall\, x \in \mathfrak{g}$$

Then $j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is a Lie alg. hom.

Then $(\mathcal{U}(\mathfrak{g}), j)$ satisfies the universal property.

<u>Thm:</u> (PBW Theorem):

$j : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is a 1-1 map.