Cartan matrix $A = (a_{ij})_{n\times n}$ (GCM)

- $a_{ii} = 2$
- $a_{ij} \le 0$, $i \neq j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, $i \neq j$

Assume $A = (a_{ij})$ is symmetrizable.

- $\exists$ nonsingular $\mathrm{diag}(s_1, s_2, \dots, s_n) = D$ s.t. DA is symmetric.
  ($\Rightarrow s_i a_{ij} = s_j a_{ji}$)

Suppose $A = (a_{ij})_{n\times n}$ is an indecomposable symmetrizable GCM.

Note: The Cartan matrix associated with a finite dim'l simple Lie alg. is an indecomposable symmetrizable matrix.

Thm: $A=(a_{ij})_{n\times n}$ indecomposable symmetrizable GCM. Then one of the following three conditions hold.

(1) A positive definite (finite)
($\Rightarrow \exists\, \delta>0$ s.t. $A\delta>0$)

(2) $\operatorname{corank}(A)=1$ and $\exists\, \delta>0$ such that $A\delta=0$ (affine)

(3) A is neither finite nor affine.
($\Rightarrow \exists\, \delta>0$ such that $A\delta<0$)
(indefinite)

Associate the Lie alg. $\mathfrak{g}=\mathfrak{g}(A)$ with $A=(a_{ij})_{n\times n}$ indecomp. symmetrizable GCM as follows.

- $\mathfrak{h}$ abelian Lie alg. of $\dim = n+\operatorname{corank}(A)$
- $\Pi=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}\subset\mathfrak{h}^*$ lin. indep.
- $\Pi^\vee=\{h_1,h_2,\cdots,h_n\}\subset\mathfrak{h}$ lin. indep.
s.t. $\alpha_j(h_i)=a_{ij}$

- $\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i \mid 1 \le i \le n \rangle$ satisfying the following conditions.

(1) $[h, h'] = 0 \;\; \forall\, h, h' \in \mathfrak{h}$

(2) $[h, e_i] = \alpha_i(h) e_i \;\; \forall\, h \in \mathfrak{h} \;\; \forall\, i$

(3) $[h, f_i] = -\alpha_i(h) f_i \;\; \forall\, h \in \mathfrak{h} \;\; \forall\, i$

(4) $[e_i, f_j] = \delta_{ij} h_i \;\; \forall\, i, j$

(5) $(\mathrm{ad}\, e_i)^{-a_{ij}+1} e_j = 0, \;\; \forall\, i \ne j$

(6) $(\mathrm{ad}\, f_i)^{-a_{ij}+1} f_j = 0, \;\; \forall\, i \ne j$.

<u>Ex(1)</u> $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ affine GCM

$\delta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \;\; A\delta = 0.$

$\mathfrak{g} = \mathfrak{g}(A)$ affine Lie algebra

$\mathfrak{h} = \mathrm{span}\{h_0, h_1, d\}$

$\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i \mid i = 0, 1 \rangle$

$\dim \mathfrak{g} = \infty$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad sl(2) = \operatorname{span}\{e, f, h\}$$

Consider the Lie algebra

$$L = sl(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with the bracket given by

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + \delta_{i+j,0}\, i\, tr(xy) c$$

$$[c, L] = 0$$

$$[d, x \otimes t^i] = i(x \otimes t^i) \quad \left(\Rightarrow d = 1 \otimes t\frac{d}{dt}\right)$$

$$[d, d] = 0$$

Define

$$e_1 = e \otimes 1, \quad f_1 = f \otimes 1, \quad h_1 = h \otimes 1$$

$$e_0 = f \otimes t, \quad f_0 = e \otimes t^{-1}, \quad h_0 = -h \otimes 1 + c$$

Note: $c = h_0 + h_1$.

H Cartan subalgebra of L is

$$H = \operatorname{span}\{h_0, h_1, d\}$$

$$\Delta = \{\alpha_1 + k\delta,\ \alpha_0 + k\delta \mid k \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}_{\neq 0}\}$$

$$= \Delta^+ \cup \Delta^-$$

$$\Delta^+ = \{\alpha_1 + k\delta,\ \alpha_0 + k\delta \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{k\delta \mid k > 0\}$$

$$\Delta = \Delta_{re} \cup \Delta_{im}$$

real imaginary

$$\Delta_{re} = \{\alpha_1 + k\delta, \alpha_0 + k\delta \mid k \in \mathbb{Z}\}$$

$$\Delta_{im} = \{k\delta \mid k \in \mathbb{Z}_{\neq 0}\}$$

Weyl group:

$$W = \langle r_0, r_1 \rangle \subset Aut(\mathfrak{h}^*)$$

$$r_i(\beta) = \beta - \underbrace{\langle \alpha_i, \beta \rangle}_{=\beta(h_i)} \alpha_i, \quad i = 0, 1$$

$$(r_0 r_1)(\alpha_0) = r_0(\alpha_0 - \alpha_0(h_1)\alpha_1)$$

$$= r_0(\alpha_0 + 2\alpha_1)$$

$$= -\alpha_0 + 2 r_0(\alpha_1)$$

$$= -\alpha_0 + 2(\alpha_1 - \alpha_1(h_0)\alpha_0)$$

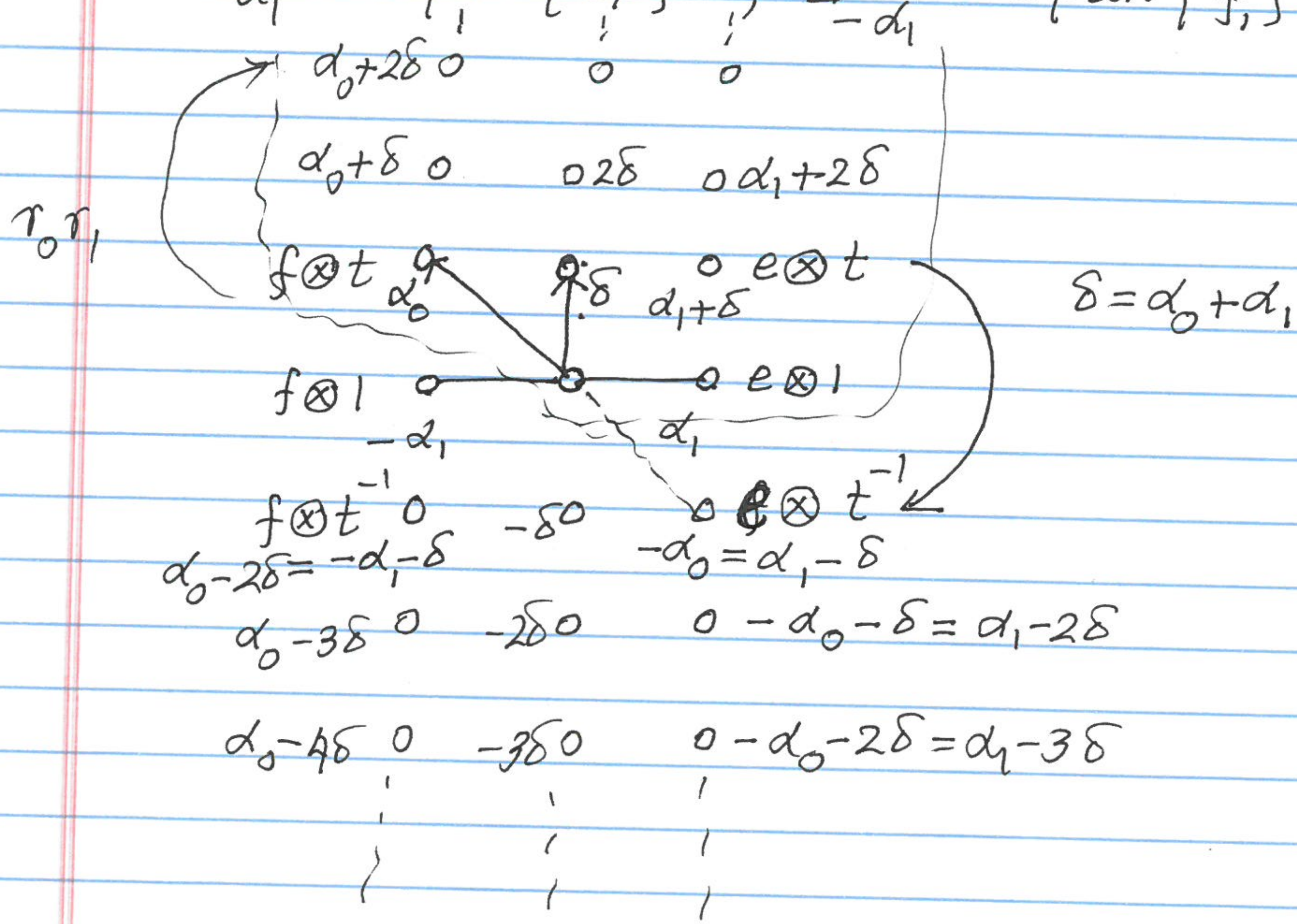
$$= -\alpha_0 + 2\alpha_1 + 4\alpha_0 = 3\alpha_0 + 2\alpha_1 = \alpha_0 + 2\delta$$

$$[e_0, f_0] = [f\otimes t, e\otimes t^{-1}]$$
$$= [f, e]\otimes 1 + tr(fe)c$$
$$= -h\otimes 1 + c = h_0 .$$

$\Pi = \{\alpha_0, \alpha_1\}$

$L_{\alpha_0} = span\{e_0\}$, $L_{-\alpha_0} = span\{f_0\}$

$L_{\alpha_1} = span\{e_1\}$, $L_{-\alpha_1} = span\{f_1\}$



$$r_0(\delta) = r_0(\alpha_0+\alpha_1) = -\alpha_0 + (\alpha_1 - \alpha_1(h_0)\alpha_0)$$
$$= -\alpha_0 + \alpha_1 + 2\alpha_0 = \alpha_0+\alpha_1 = \delta$$

Similarly, $r_1(\delta) = \delta$

$\Longrightarrow w \in W$, $\alpha \in \Delta_{im} = \{k\delta \mid k \in \mathbb{Z}_{\neq 0}\}$

$\Longrightarrow w\alpha = \alpha$.

Observation :

(1) $\forall\ \alpha \in \Delta_{re}$, $\exists\ i = 0$ or 1 and $w \in W$ such that $w\alpha_i = \alpha$

Indeed, a root $\alpha \in \Delta$ is called real if $\exists\ w \in W$ such that $w\alpha_i = \alpha$ for some $i = 0, 1$.

$\alpha \in \Delta$ is imaginary if it is not real.

(2) $\alpha \in \Delta_{re}$ and $\alpha = w\alpha_i$

$\Longrightarrow \dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w\alpha_i} = \dim \mathfrak{g}_{\alpha_i} = 1$

(3) ~~$\alpha \in \Delta_{im}$~~ $\delta \in \Delta_{im}$

$\mathfrak{g}_\delta = \mathrm{span}\{h \otimes t\} \Rightarrow \dim \mathfrak{g}_\delta = 1$

In general, $\dim \mathfrak{g}_\delta = \mathrm{rank}(A)$.

(4) $W = \{ (r_0 r_1)^m , r_0 (r_0 r_1)^m \mid m \in \mathbb{Z} \}$

called the infinite dihedral group