Cartan matrix $C = (c_{ij})_{l \times l}$

- $c_{ii} = 2$
- $c_{ij} \le 0, \quad i \ne j$
- $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$
- C is pos. definite

Serre's Thm (~1965) (Humphrey, Thm 8.3)

Fix root system (Δ, Π), $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$

$C = (c_{ij})_{l \times l} = (\langle \alpha_i, \alpha_j \rangle)_{l \times l}$ Cartan matrix.

Then the finite dim'l semisimple Lie algebra L with Cartan matrix C is generated by $3l$ elements $\{e_i, f_i, h_i \mid 1 \le i \le l\}$ subject to the following relations:

Chevalley relations

(1) $[h_i, h_j] = 0 \quad \forall i, j$

(2) $[h_i, e_j] = c_{ij} e_j \quad \forall i, j$

(3) $[h_i, f_j] = -c_{ij} f_j \quad \forall i, j$

(4) $[e_i, f_j] = \delta_{ij} h_i \quad \forall i, j$

Serre relations

(5) $(\mathrm{ad}_{e_i})^{-c_{ij}+1} e_j = 0 \quad \forall i \ne j$

(6) $(\mathrm{ad} f_i)^{-c_{ij}+1} f_j = 0 \quad \forall i \ne j$

Ex (1) $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{2\times 2}$ Cartan matrix

generators : $e_1, e_2, f_1, f_2, h_1, h_2$

$[e_1, e_2]$: $ad^2_{e_1}(e_2) = 0 \Rightarrow [e_1,[e_1,e_2]] = 0$

$[f_1, f_2]$: $ad^2_{f_1}(f_2) = 0 \Rightarrow [f_1,[f_1,f_2]] = 0$

$\Rightarrow L = \text{span}\{e_1, e_2, f_1, f_2, h_1, h_2, [e_1,e_2], [f_1,f_2]\}$

$L \cong sl(3, \mathbb{C})$

$e_1 \longmapsto E_{12}$, $e_2 \longmapsto E_{23}$

$f_1 \longmapsto E_{21}$, $f_2 \longmapsto E_{32}$

$h_1 \longmapsto E_{11} - E_{22}$, $h_2 \longmapsto E_{22} - E_{33}$

$[e_1, e_2] \longmapsto E_{13}$, $[f_1, f_2] \longmapsto -E_{31}$

$H = \text{span}\{h_1, h_2\}$ max'l toral subalg.

$\alpha_1, \alpha_2 \in H^*$

$\alpha_1(h_1) = 2$, $\alpha_1(h_2) = -1$

$\alpha_2(h_1) = -1$, $\alpha_2(h_2) = 2$

$\Rightarrow [h_1, e_1] = \alpha_1(h_1) e_1$, $[h_i, e_j] = \alpha_j(h_i) e_j$

Generalized Cartan Matrix :

$$C = (c_{ij})_{\ell \times \ell} \quad \text{satisfying}$$

- $c_{ii} = 2$
- $c_{ij} \leq 0 \qquad i \neq j$
- $c_{ij} = 0 \Longleftrightarrow c_{ji} = 0$

Ex(2) $\quad C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}_{2 \times 2}$

generators: $e_0, e_1, f_0, f_1, h_0, h_1$

$(ad_{e_0})^3 e_1 = 0 = (ad_{e_1})^3 e_0$

$(ad_{f_0})^3 f_1 = 0 = (ad_{f_1})^3 f_0$

$[e_0, e_1], \quad [e_0, [e_0, e_1]]$

$[f_0, f_1], \quad [f_0, [f_0, f_1]]$

$[e_1, [e_0, [e_0, e_1]]], \quad [e_1, [e_1, [e_0, [e_0, e_1]]]]$

$\vdots$

$\Rightarrow L$ is infinite dim'l.

$H = \text{span}\{h_0, h_1\}$, $\alpha_1, \alpha_2 \in H^*$

$\cancel{\alpha_1(h_1)}$ $\alpha_j(h_i) = c_{ij}$

$\alpha_0(h_0) = 2$, $\alpha_0(h_1) = -2$

$\alpha_1(h_0) = -2$, $\alpha_1(h_1) = 2$

$(\alpha_0 + \alpha_1)(h_1) = 0 = (\alpha_1 + \alpha_2)(h_2)$

$\Rightarrow \alpha_0 + \alpha_1 = 0$

$d \in \text{Der}\, L$

$d(h_0) = 0$, $d(h_1) = 0$

$d(e_0) = e_0$, $d(e_1) = 0$

$d(f_0) = -f_0$, $d(f_1) = 0$

$$d(h_0) = d([e_0, f_0]) = [d(e_0), f_0] + [e_0, d(f_0)]$$
$$= h_0 - h_0 = 0.$$

d is called a degree derivation.

$H^e = H \ltimes \mathbb{C}d = \text{span}\{h_0, h_1, d\}$

$0 \neq \alpha \in (H^e)^*$ is a <u>root</u> if

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \;\forall h \in H^e\} \neq 0$$

$\alpha_0, \alpha_1 \in (H^e)^*$

$\alpha_0(h_0) = 2 \;,\; \alpha_0(h_1) = -2$

$\alpha_1(h_0) = -2 \;,\; \alpha_1(h_1) = 2$

$\alpha_0(d) = 1 \;,\; \alpha_1(d) = 0$

$\alpha_0 + \alpha_1 \in (H^e)^* \;,\; \alpha_0 + \alpha_1 \neq 0$

since $(\alpha_0 + \alpha_1)(d) = 1$

$[h_0, e_0] = 2e_0 \;,\; [h_0, e_1] = -2e_1$
$[h_1, e_0] = -2e_0 \;,\; [h_1, e_1] = 2e_1$

$[h_0, f_0] = -2f_0 \;,\; [h_0, f_1] = 2f_1$

$[h_1, f_0] = +2f_0 \;,\; [h_1, f_1] = -2f_1$

$H = \text{span}\{h_0, h_1\}$

$\{ad_{h_0}, ad_{h_1}\}$ $\quad ad_{h_0}(e_0) = 2e_0$

$ad_{h_1}(e_0) = -2e_0$

$ad_{h_0}(f_1) = 2f_1$

$ad_{h_1}(f_1) = -2f_1$

$\Longrightarrow \dim(\alpha_0\text{-root space}) = 2 \quad \cancel{=}$

$$ad_d(e_0) = [d, e_0] = d(e_0) = 1e_0$$

$$ad_d(f_1) = [d, f_1] = d(f_1) = 0$$

$$\Rightarrow e_0 \in L_{\alpha_0} \text{ but } f_1 \notin L_{\alpha_0}$$

Works!

$C = (c_{ij})_{\ell \times \ell}$ generalized Cartan matrix. We define the Cartan datum:

$$(H, \Pi, \check{\Pi}), \quad \Pi \subset H^*, \quad \check{\Pi} \in H$$

where

- H is a (maximal) abelian Lie alg. with $\dim H = \ell + \operatorname{corank}(C)$
- $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ lin. indep. simple roots
- $\check{\Pi} = \{h_1, h_2, \cdots, h_\ell\}$ lin. indep. simple coroots.

Define the Lie algebra L associated with $(C, H, \Pi, \check{\Pi})$ to be the Lie alg. generated by $H \cup \{e_i, f_i \mid 1 \le i \le \ell\}$

satisfying the six relations.

Further assume that the GCM is symmetrizable; i.e.

$\exists$ a nonsingular diagonal matrix D such that DC is symmetric.