(Δ, Π) irreducible root system

$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ simple roots.

$$\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$$

Cartan Matrix:

$$C = (c_{ij})_{\ell \times \ell}, \quad c_{ij} = \langle \alpha_i, \alpha_j \rangle$$

$\Rightarrow$
(1) $c_{ii} = 2$
(2) $c_{ij} \leq 0, \ i \neq j$
(3) $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$
(4) C is pos. definite

Dynkin Diagram: Γ assoc. with C.

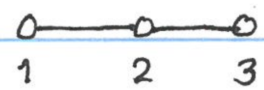
- has ℓ nodes
- For $i \neq j$, the ith and jth node are connected by $\underbrace{\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle}_{= 0, 1, 2, \text{ or } 3}$ no. of edges.
- If $|\langle \alpha_i, \alpha_j \rangle| \neq |\langle \alpha_j, \alpha_i \rangle|$, then draw an arrow pointing to the ith node if $|\langle \alpha_i, \alpha_j \rangle| > 1$.

Recall: (Δ, Π) irred. $\Rightarrow$ C is indecomposable

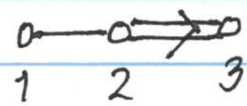
$\Rightarrow$ Γ is connected.

<u>Remark</u>: The Cartan matrix is completely determined by the Dynkin diagram Γ and conversely.

<u>Ex(1)</u> $$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}_{3\times 3} \qquad \mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$$

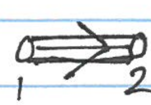
$\Gamma =$ 

<u>Ex(2)</u>

Γ:  $\qquad \mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}_{3\times 3}$$

<u>Ex(3)</u> $$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}_{2\times 2} \qquad \mathfrak{g} = G_2$$

Γ: 

Remark: The irred. root system (Δ, Π) is completely determined by the Cartan matrix $C = (\langle \alpha_i, \alpha_j \rangle)_{\ell \times \ell}$ (hence by the Dynkin diagram).

Example: Consider the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}_{3\times 3}$$

$\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$. Want to determine the set of all roots Δ. Since

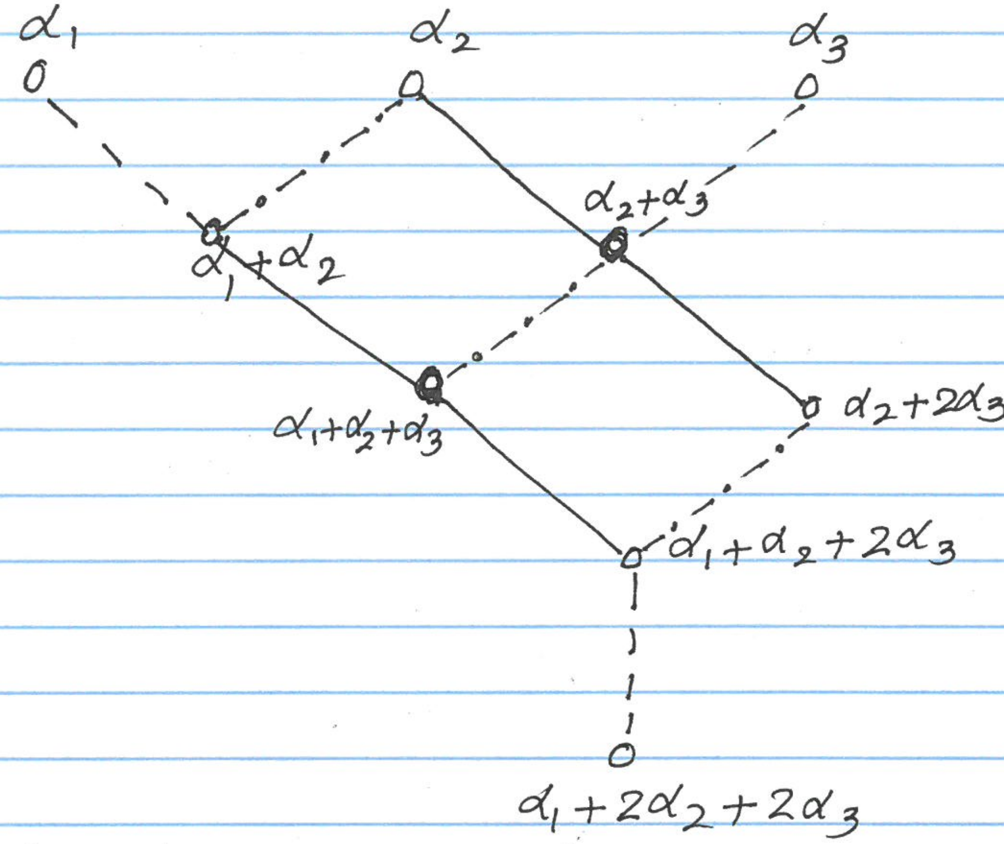
$$\Delta = \Delta^+ \cup \Delta^- \text{ and } \Delta^- = -\Delta^+$$

it is enough to determine the set of positive roots Δ^+.

α_1 - string: —·—·—·—

α_2 - string: — — — ——

α_3 - string: ————————



(See details in page 139-140)

$$\therefore \quad \Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_2+2\alpha_3, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+2\alpha_2+2\alpha_3\}$$

$$\Delta = \Delta^+ \cup -\Delta^+$$

α_2-string through α_1 : $\alpha_1, \alpha_1+\alpha_2, \cdots, \alpha_1+q\alpha_2$

$r=0 \Rightarrow -q = \langle \alpha_2, \alpha_1 \rangle = -1 \Rightarrow q=1$

α_1-string thro' α_2 : $\alpha_2, \alpha_2+\alpha_1, \cdots, \alpha_2+q\alpha_1$

$r=0 \Rightarrow -q = \langle \alpha_1, \alpha_2 \rangle = -1 \Rightarrow q=1$

α_3-string thro' α_2 : $\alpha_2, \alpha_2+\alpha_3, \cdots, \alpha_2+q\alpha_3$

$-q = \langle \alpha_3, \alpha_2 \rangle = -2 \Rightarrow q=2$

α_2-string thro' α_3 : $\alpha_3, \cdots, \alpha_3+q\alpha_2$

$-q = \langle \alpha_2, \alpha_3 \rangle = -1 \Rightarrow q=1$

α_3-string thro' $\alpha_1+\alpha_2$: $(\alpha_1+\alpha_2), \cdots, (\alpha_1+\alpha_2)+q\alpha_3$

$$-q = \langle \alpha_3, \alpha_1+\alpha_2 \rangle = \langle \alpha_3, \alpha_1 \rangle + \langle \alpha_3, \alpha_2 \rangle = -2$$

$\Rightarrow q = 2$

α_1-string thro' $\alpha_2+\alpha_3$; $(\alpha_2+\alpha_3), \cdots, (\alpha_2+\alpha_3)+q\alpha_1$

$$-q = \langle \alpha_1, \alpha_2+\alpha_3 \rangle = \langle \alpha_1, \alpha_2 \rangle + \langle \alpha_1, \alpha_3 \rangle = -1$$

$\Rightarrow q=1$

α_2-string thro' $\alpha_1+\alpha_2+\alpha_3$:

$-q = \langle \alpha_2, \alpha_1+\alpha_2+\alpha_3 \rangle = -1+2-1=0$

α_1-string thro' $\alpha_2+2\alpha_3$: $-q = \langle \alpha_1, \alpha_2+2\alpha_3 \rangle = -1$

$\Rightarrow q=1$

α_2-string thro' $\alpha_1+\alpha_2+2\alpha_3$:

$$-q = \langle \alpha_2, \alpha_1+\alpha_2+2\alpha_3 \rangle = -1+2-2 = -1$$

$$\Rightarrow q = 1$$

Classification of Irreducible root systems

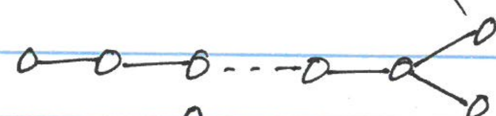
(Humphrey: Thm 11.4)

If (Δ, Π), $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ is an irred. root system of rank ℓ, then its Dynkin diagram is one of the following:

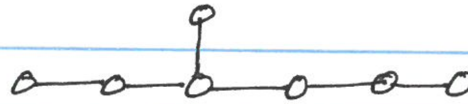
(1) $sl(\ell+1, \mathbb{C})$:  : A_ℓ $(\ell \geq 1)$

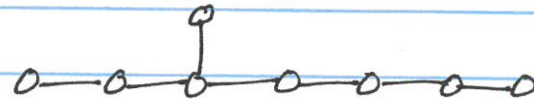
(2) $so(2\ell+1, \mathbb{C})$:  : B_ℓ $(\ell \geq 2)$

(3) $sp(2\ell, \mathbb{C})$:  : C_ℓ $(\ell \geq 3)$

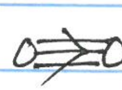
(4) $so(2\ell, \mathbb{C})$:  : D_ℓ $(\ell \geq 4)$

(5)  : E_6

(6)  : E_7

(7)  : E_8

(8)  : F_4

(9)  : G_2

E_6, E_7, E_8, F_4, G_2: exceptional cases