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$\Delta \subseteq E$ root system of rank ℓ.

Fix a base $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$, simple roots. Recall

- Π is a basis for E, and
- $\alpha \in \Delta \Rightarrow \alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ where

all $k_i \in \mathbb{Z}_{\geq 0}$ or all $k_i \in \mathbb{Z}_{\leq 0}$.

$$\Delta^+ = \{\beta \in \Delta \mid \beta = \sum_{i=1}^{\ell} k_i \alpha_i,\ k_i \in \mathbb{Z}_{\geq 0}\}$$

$$\Delta^- = \{\beta \in \Delta \mid \beta = \sum_{i=1}^{\ell} k_i \alpha_i,\ k_i \in \mathbb{Z}_{\leq 0}\}$$

Then $\Delta = \Delta^+ \cup \Delta^-$, $\Delta^- = -\Delta^+$

Δ^+ = set of positive roots

Δ^- = set of negative roots.

For $\beta \in \Delta^+$, $\beta = \sum_{i=1}^{\ell} k_i \alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$

define $ht(\beta) = \sum_{i=1}^{\ell} k_i$, called height of β.

Recall $(\ ,\ )$ symm. pos. definite bilinear form on E and

$(\alpha_i, \alpha_j) < 0$, $i \neq j$.

<u>Prop</u>: $\beta \in \Delta^+$, $ht(\beta) > 1$.

$\Rightarrow \exists\ \alpha_j \in \Pi$ such that $\beta - \alpha_j \in \Delta^+$.

<u>Pf</u>: Suppose not. Then $\beta - \alpha_i$ not a root for any $1 \le i \le l$.

$\Rightarrow (\beta, \alpha_i) \le 0$, $1 \le i \le l$.

$$(\beta, \beta) = \left(\beta, \sum_{i=1}^{l} k_i \alpha_i\right) \quad , \quad k_i \in \mathbb{Z}_{\ge 0}$$

$$= \sum_{i=1}^{l} k_i (\beta, \alpha_i) \le 0$$

But $(\beta, \beta) \ge 0$ since $(\,,\,)$ pos. def.

Hence $(\beta, \beta) = 0 \Rightarrow \beta = 0$ which is a contradiction. //

For the root system (Δ, Π) define

$$C = \left(\langle \alpha_i, \alpha_j \rangle\right)_{1 \le i,j \le l} = (c_{ij}) , \quad \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

called the Cartan Matrix.

Then C has the following properties:

(1) $c_{ii} = 2$

(2) $c_{ij} \le 0$, $i \ne j$

(3) $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$, $i \ne j$

(4) C is positive definite.

<u>Defn:</u> A subset S of Δ is reducible if $S = S_1 \cup S_2$, $S_i \neq \phi$, $i=1,2$ and $(S_1, S_2) = 0$.

$(,)$ pos. definite $\Rightarrow S_1 \cap S_2 = \phi$.

We say S is irreducible if it is not reducible.

<u>Thm:</u> Π is reducible $\Leftrightarrow$ Δ reducible.

<u>Pf:</u> ($\Leftarrow$) Suppose Δ is reducible.

$\Rightarrow \Delta = \Delta_1 \cup \Delta_2$, $\Delta_i \neq \phi$, $(\Delta_1, \Delta_2) = 0$.

Then $\Pi = \underbrace{(\Pi \cap \Delta_1)}_{\Pi_1} \cup \underbrace{(\Pi \cap \Delta_2)}_{\Pi_2}$

Enough to show $\Pi_i \neq \phi$, $i = 1,2$.

Suppose $\Pi_1 = \phi = \Pi \cap \Delta_1$

$\Rightarrow \Pi \subset \Delta_2$

$\Rightarrow \forall \beta \in \Delta_1$ we have $(\beta, \Pi) = 0$

$\Rightarrow (\beta, E) = 0 \Rightarrow (\beta, \beta) = 0 \Rightarrow \beta = 0$ which is a contradiction since $\Delta_1 \neq \phi$.

$\Rightarrow \Pi_1 \neq \phi$. Similarly $\Pi_2 \neq \phi$.

($\Longrightarrow$) Suppose Π is reducible.

$\Longrightarrow \Pi = \Pi_1 \cup \Pi_2$, $\Pi_i \neq \phi$, $i=1,2$ and $(\Pi_1, \Pi_2) = 0$.

Set $\Delta_i = \text{span}_{\mathbb{R}}(\Pi_i) \cap \Delta$, $i=1,2$.

Since $\Delta_i \supset \Pi_i \Rightarrow \Delta_i \neq \phi$ $i=1,2$.

$(\Pi_1, \Pi_2) = 0 \Rightarrow (\Delta_1, \Delta_2) = 0$.

Left to show $\Delta_1 \cup \Delta_2 = \Delta = \Delta^+ \cup \Delta^-$

Enough to show $\forall\ \beta \in \Delta^+$, $\beta \in \Delta_1 \cup \Delta_2$.

Use induction on $ht(\beta)$.

If $ht(\beta)=1$, then $\beta \in \Pi$. Hence $\beta \in \Delta_i$ for some $i=1,2$.

Suppose $ht(\beta) > 1$ and assume for any $\gamma \in \Delta^+$ with $ht(\gamma) < ht(\beta)$ we have $\gamma \in \Delta_i$ for some $i=1,2$.

$ht(\beta) > 1 \Rightarrow \beta \notin \Pi$.

$\Rightarrow \beta = \gamma_1 + \gamma_2$, $\gamma_1, \gamma_2 \in \Delta^+$ (by Prop.)

Since $ht(\gamma_i) < ht(\beta)$, $i = 1, 2$

$\Rightarrow \gamma_i \in \Delta_j$ for some $j = 1, 2$.

If $\gamma_1, \gamma_2 \in \Delta_i \Rightarrow \gamma_1 + \gamma_2 \in \Delta_i \Rightarrow \beta \in \Delta_i$.

Suppose $\gamma_1 \in \Delta_1$, $\gamma_2 \in \Delta_2$.

$(\Delta_1, \Delta_2) = 0 \Rightarrow (\gamma_1, \gamma_2) = 0$.

Suppose $\gamma_i = \sum_{j=1}^{\ell} k_{ji} \alpha_j$, $i = 1, 2$, $k_{ji} \in \mathbb{Z}_{\geq 0}$
$\alpha_j \in \Pi_i$

$$\begin{aligned}
\Delta \ni \sigma_{\gamma_1}(\beta) &= \sigma_{\gamma_1}(\gamma_1 + \gamma_2) \\
&= \sigma_{\gamma_1}(\gamma_1) + \sigma_{\gamma_1}(\gamma_2) \\
&= -\gamma_1 + \gamma_2 - \underbrace{\langle \gamma_1, \gamma_2 \rangle}_{=0} \gamma_1 = -\gamma_1 + \gamma_2
\end{aligned}$$

which is a contradiction.

$\Rightarrow \gamma_1, \gamma_2 \in \Delta_i$ for some $i = 1, 2$

$\Rightarrow \beta = \gamma_1 + \gamma_2 \in \Delta_i$ for some $i = 1, 2$. //

Hence it is sufficient to classify all irreducible root systems.

Fix an irreducible root system of rank ℓ

(Δ, Π), $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ simple roots.

$$C = (\langle \alpha_i, \alpha_j \rangle)_{\ell \times \ell} \quad \text{Cartan matrix}$$

Recall an $\ell \times \ell$ matrix $A = (a_{ij})_{\ell \times \ell}$ is decomposable if $\exists$ a permutation $\tau \in S_\ell$ such that

$$(a_{\tau(i)\tau(j)})_{\ell \times \ell} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

It is indecomposable if A is not decomposable.

<u>Remark:</u> (Δ, Π) is irreducible $\Longleftrightarrow$ $C = (\langle \alpha_i, \alpha_j \rangle)_{\ell \times \ell}$ is indecomposable.