$E \supset \Delta$, $\dim E = \ell$.

Lemma: $\alpha, \beta \in \Delta$, $\beta \neq \pm\alpha$

(i) $(\beta,\alpha) < 0 \Longrightarrow \beta + \alpha \in \Delta$ (i.e. θ obtuse)

(ii) $(\beta,\alpha) > 0 \Longrightarrow \beta - \alpha \in \Delta$ (i.e. θ acute)

Pf (i) Assume $\overset{(\alpha,\beta)}{(\beta,\alpha)} < 0$

$$\Longrightarrow \langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} < 0$$

$$\& \quad \langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\beta,\beta)} < 0$$

$$\Longrightarrow \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \{\cancel{0}, 1, 2, 3, \cancel{4}\}$$

$(\beta,\alpha) \neq 0$ $\qquad$ $\beta \neq \pm\alpha$

$$\Longrightarrow \langle \alpha, \beta \rangle = -1 \quad \text{or} \quad \langle \beta, \alpha \rangle = -1$$

If $\langle \beta, \alpha \rangle = -1$, Then

$\cancel{\sigma_\beta(\alpha) = \alpha - \langle \beta, \alpha \rangle \alpha}$

$\cancel{\sigma_\alpha(\beta) = \beta -}$

$$\sigma_\beta(\alpha) = \alpha - \langle \beta, \alpha \rangle \beta = \alpha + \beta \in \Delta$$

If $\langle \alpha, \beta \rangle = -1$

$$\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha = \beta + \alpha \in \Delta \quad //$$

<u>Thm:</u> $\alpha, \beta \in \Delta$, $\beta \neq \pm\alpha$

Then $\left|\{\beta + i\alpha \in \Delta \mid i \in \mathbb{Z}\}\right| \leq 4$

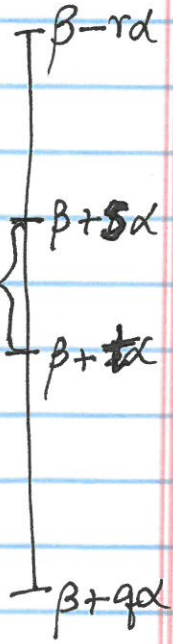
and the α-string through β is unbroken.

<u>Pf:</u> Choose the largest nonnegative integers r and q such that

$$\beta - r\alpha \in \Delta \text{ , and } \beta + q\alpha \in \Delta$$

Suppose the string

$$\beta - r\alpha, \cdots, \beta, \cdots, \beta + q\alpha$$



is broken. Choose $t, s \in \mathbb{Z}$, $t > s$ such that, t is largest and

$$\beta + t\alpha \in \Delta \text{ , } \beta + (t-1)\alpha \notin \Delta$$

and s is smallest and

$$\beta + s\alpha \in \Delta \text{ , } \beta + (s+1)\alpha \notin \Delta$$

By Lemma, we have

$$(\beta + t\alpha, \alpha) \leq 0 \text{ , } (\beta + s\alpha, \alpha) \geq 0$$

$$\Rightarrow (\beta + t\alpha, \alpha) \leq (\beta + s\alpha, \alpha)$$

$$\Rightarrow \cancel{(\beta,\alpha)} + t(\alpha,\alpha) \le \cancel{(\beta,\alpha)} + s(\alpha,\alpha)$$

$$\Rightarrow \quad t(\alpha,\alpha) \le s(\alpha,\alpha) \Rightarrow t \le s$$

which is a contradiction, since $t > s$.

$\Rightarrow$ the ~~β~~ α-string through β is unbroken.

Left to show that length is atmost 4.

$$\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha \in \Delta$$

Apply σ_α to this string

$$\sigma_\alpha(\beta - r\alpha), \ldots, \sigma_\alpha(\beta), \ldots, \sigma_\alpha(\beta + q\alpha) \in \Delta$$

$$\sigma_\alpha(\beta - r\alpha) = \beta - r\alpha - \langle\alpha, \beta - r\alpha\rangle\alpha = \sigma_\alpha(\beta) + r\alpha$$

$$\sigma_\alpha(\beta) = \beta - \langle\alpha,\beta\rangle\alpha$$

$$\sigma_\alpha(\beta + q\alpha) = \beta + q\alpha - \langle\alpha, \beta + q\alpha\rangle\alpha = \underline{\beta} + q\alpha - \underline{\langle\alpha,\beta\rangle\alpha} - q\underbrace{\langle\alpha,\alpha\rangle}_{=2}\alpha = \sigma_\alpha(\beta) - q\alpha$$

~~$\Rightarrow$ the~~

$\Rightarrow$ the string has reversed.

$$\Rightarrow \beta - r\alpha = \sigma_\alpha(\beta) - q\alpha = \beta - \langle\alpha,\beta\rangle\alpha - q\alpha$$

$$\Rightarrow \quad r = q + \langle\alpha,\beta\rangle \Rightarrow r - q = \langle\alpha,\beta\rangle$$

Case: $q = 0$ (i.e. $\beta + \alpha \notin \Delta$)

$\Rightarrow$ $(\beta, \alpha) \geq 0$ and the α-string through β looks like

$$\beta - r\alpha, \ldots, \beta$$

and $r = \langle \alpha, \beta \rangle$

$(\beta, \alpha) \geq 0 \Rightarrow \langle \alpha, \beta \rangle \geq 0$

Recall $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3, \cancel{4}\}$
$\beta \neq \pm\alpha$

$\Rightarrow \underbrace{\langle \alpha, \beta \rangle}_{= r} \leq 3$

$\Rightarrow$ α-string through β has length atmost 4.

Case: $q > 0$

$$\underbrace{\beta - r\alpha}_{= \gamma - (r+q)\alpha}, \ldots, \beta, \ldots, \underbrace{\beta + q\alpha}_{= \gamma} \in \Delta$$

which the α-string through γ with $q = 0$

$\Rightarrow r + q \leq 3$ (by previous case)

$\Rightarrow$ the α-string through β is of length at most 4. //

Defn: Let Δ be a root system of rank l (i.e. $\dim E = l$).
A subset Π of Δ is a base if
(1) Π is a basis for E

(2) For $\beta \in \Delta$, $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ (unique)
$\Rightarrow$ $c_\alpha \in \mathbb{Z}_{\geq 0}$ $\forall \alpha \in \Pi$ or
$c_\alpha \in \mathbb{Z}_{\leq 0}$ $\forall \alpha \in \Pi$

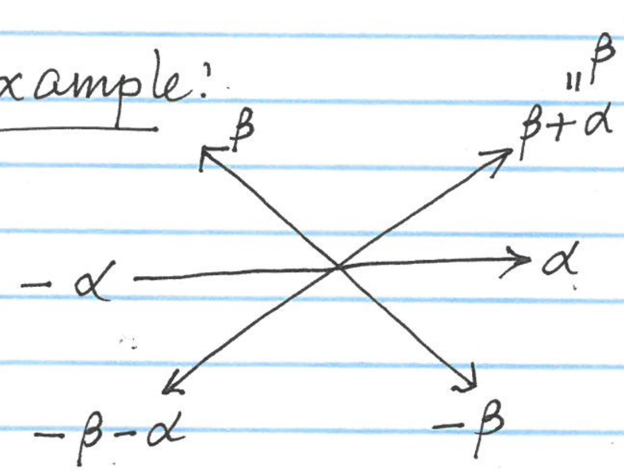
In such case, we say that Π is a set of simple roots.

Lemma: Δ be a root system with base Π. Let $\alpha, \beta \in \Pi$. Then

$(\alpha, \beta) \leq 0$ ($\Leftrightarrow$ the angle θ between α & β is obtuse)

Pf: Suppose $(\alpha, \beta) > 0$. Then $\beta - \alpha \in \Delta$ which is a contradiction to the defn of Π. //

Example:



$\{\alpha, \beta\} = \Pi$

but $\{\beta+\alpha, \alpha\}$ is not a base.

Existence of base Π for a root system Δ.

For $\alpha \in \Delta$, $P_\alpha = (\mathbb{R}\alpha)^\perp \subseteq E$

$$\bigcup_{\alpha \in \Delta} P_\alpha \neq E$$

Choose $\rho \in E \setminus \bigcup_{\alpha \in \Delta} P_\alpha$

Define $\Delta^+(\rho) = \{\alpha \in \Delta \mid (\rho, \alpha) > 0\}$

$\Delta^-(\rho) = \{\alpha \in \Delta \mid (\rho, \alpha) < 0\}$

Since $\rho \notin \bigcup_{\alpha \in \Delta} P_\alpha$, $(\rho, \alpha) \neq 0 \ \forall \alpha \in \Delta$

$\Longrightarrow \Delta^+(\rho) \cup \Delta^-(\rho) = \Delta$

Note $\Delta^-(\rho) = -\Delta^+(\rho)$

Say $\alpha \in \Delta^+(\rho)$ is simple if

$\alpha \neq \beta_1 + \beta_2$ for any $\beta_1, \beta_2 \in \Delta^+(\rho)$

Take ~~$\Delta(\rho) = \mathfrak{se}$~~

$\Pi(\rho)$ = set of simple roots in $\Delta^+(\rho)$

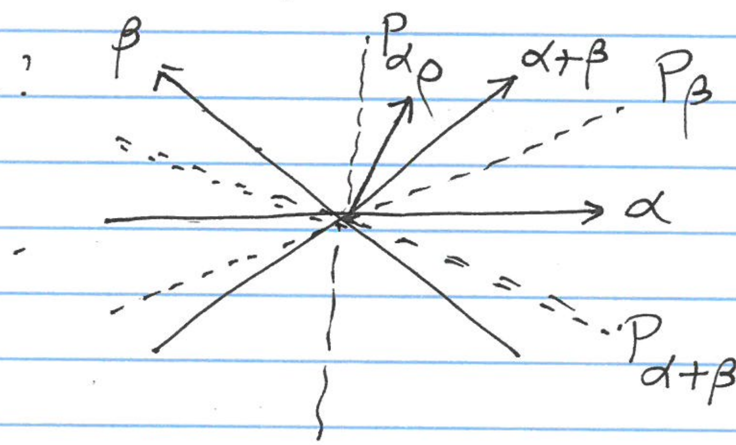
Then we have:

<u>Thm:</u> 1) $\Pi(\rho)$ is a base for Δ

and 2) Every base of Δ arise this way.

<u>Pf:</u> (Read Humphrey § 10.1)

<u>Example:</u>



$\Pi(\rho) = \{\alpha, \beta\}$ set of simple roots.