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Δ root system

$\Delta \subseteq \mathbb{R}^\ell$, $\mathbb{R}^\ell = \text{span}_{\mathbb{R}}\{\Delta\} = E$

$\{\alpha_1, \alpha_2, \cdots, \alpha_\ell\} \subseteq \Delta$ basis for $\mathbb{R}^\ell$

$\alpha \in \Delta$

$\sigma_\alpha : E \longrightarrow E$

$\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$, reflections

where

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

$(,) : E \times E \longrightarrow \mathbb{R}$ nondeg. sym. bil. form.

$W \subset \text{Aut} E$, $W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$

$w \in W$, $w = \sigma_\alpha \sigma_\beta \cdots \sigma_\gamma$, $\alpha, \beta, \cdots, \gamma \in \Delta$.

Know W is a finite group.

<u>Prop</u>: $w \in W \Rightarrow (w(\gamma), w(\delta)) = (\gamma, \delta)$ $\forall\ \gamma, \delta \in E$ (i.e. w is an isometry, hence W is an orthogonal group.)

Pf: Enough to show that $\forall\, \gamma, \delta \in E,\ \alpha \in \Delta$

$$(\sigma_\alpha(\gamma), \sigma_\alpha(\delta)) = (\gamma, \delta).$$

$$\sigma_\alpha(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha \ , \ \sigma_\alpha(\delta) = \delta - \langle \alpha, \delta \rangle \alpha$$

$$(\sigma_\alpha(\gamma), \sigma_\alpha(\delta)) = (\gamma, \delta) - \langle \alpha, \delta \rangle (\gamma, \alpha) - \langle \alpha, \gamma \rangle (\alpha, \delta) + \langle \alpha, \gamma \rangle \langle \alpha, \delta \rangle (\alpha, \alpha)$$

$$= (\gamma, \delta) - \frac{2(\alpha,\delta)(\gamma,\alpha)}{(\alpha,\alpha)} - \cancel{\frac{2(\alpha,\gamma)(\alpha,\delta)}{(\alpha,\alpha)}} + \cancel{\frac{2(\alpha,\gamma)}{(\alpha,\alpha)}} \frac{2(\alpha,\delta)}{\cancel{(\alpha,\alpha)}} \cancel{(\alpha,\alpha)}$$

$$= (\gamma, \delta) \qquad //$$

For $\alpha \in E$, define

$$\|\alpha\| = (\alpha, \alpha)^{1/2} \ , \quad (\alpha, \alpha) \geq 0$$

$\forall\, \alpha \in \Delta$, $(\alpha, \alpha) > 0$

$$\|\alpha\| \neq 0.$$

Cauchy-~~[illegible]~~ Schwarz Inequality:

$\forall\, \alpha, \beta \in E$, $\alpha \neq 0$, $\beta \neq 0$

$$-1 \le \underbrace{\frac{(\alpha, \beta)}{\|\alpha\|\|\beta\|}}_{\substack{\parallel \\ \cos\theta}} \le 1$$

$\cos\theta$, θ angle between α & β.

$\Rightarrow \exists!\ 0 \le \theta \le \pi$ such that

$$\cos\theta = \frac{(\alpha,\beta)}{\|\alpha\|\|\beta\|}$$

<u>Prop:</u> For $\alpha, \beta \in \Delta$, the angle θ between α & β are one of the following:

$$0, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \text{ or } \pi.$$

<u>Pf!</u> Axiom (3) $\Rightarrow \langle\alpha,\beta\rangle \in \mathbb{Z}$, $\langle\beta,\alpha\rangle \in \mathbb{Z}$

$$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\frac{2(\beta,\alpha)}{(\beta,\beta)} = \frac{4(\alpha,\beta)^2}{\|\alpha\|^2\|\beta\|^2}$$

$$= 4\cos^2\theta \in \mathbb{Z}$$

$$\Rightarrow \langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2\theta \in \{0, 1, 2, 3, 4\}$$

$$\Rightarrow \cos^2\theta \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

$$\Rightarrow \cos\theta \in \left\{0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\}$$

$$\Rightarrow \theta \in \left\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}, 0, \pi\right\} \quad //$$

Let $\alpha, \beta \in \Delta$, suppose $\|\beta\| \geq \|\alpha\|$

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3, 4\}$$

$$\frac{\langle \alpha, \beta \rangle}{\langle \beta, \alpha \rangle} = \frac{2(\alpha,\beta)/(\alpha,\alpha)}{2(\beta,\alpha)/(\beta,\beta)} = \frac{\|\beta\|^2}{\|\alpha\|^2} \geq 1$$

$$\Rightarrow \quad |\langle \alpha, \beta \rangle| \geq |\langle \beta, \alpha \rangle|$$

<u>Classify the rank 2 (i.e. $l=2$) root systems</u>

$\{\alpha, \beta\} \subseteq \Delta$, $\|\beta\| \geq \|\alpha\|$

$\{\alpha, \beta\}$ basis for $E = \mathbb{R}^2$.

Cases	$\langle \alpha,\beta \rangle \langle \beta,\alpha \rangle$	$\langle \alpha,\beta \rangle$	$\langle \beta,\alpha \rangle$	θ	$\|\beta\|^2/\|\alpha\|^2 = \frac{\langle \alpha,\beta \rangle}{\langle \beta,\alpha \rangle}$
1.	0	0	0	$\pi/2$	undetermined
2.	1	1	1	$\pi/3$	1
		-1	-1	$2\pi/3$	
3.	2	2	1	$\pi/4$	2
		-2	-1	$3\pi/4$	
4.	3	3	1	$\pi/6$	3
		-3	-1	$5\pi/6$	
0.	4	~	~	~	~

Case 0: $4\cos^2\theta = 4$

$\Rightarrow \cos^2\theta = 1 \Rightarrow \cos\theta = 1$ or -1

$\Rightarrow \theta = 0$ or π

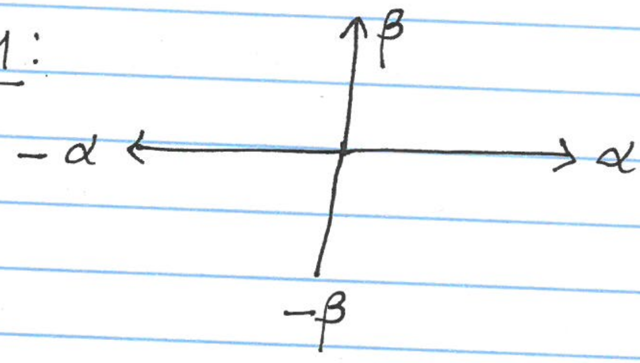
$\Rightarrow \beta = c\alpha \Rightarrow \beta = -\alpha$, $\beta \neq \alpha$

$\Rightarrow$ rank $= 1$

i.e.

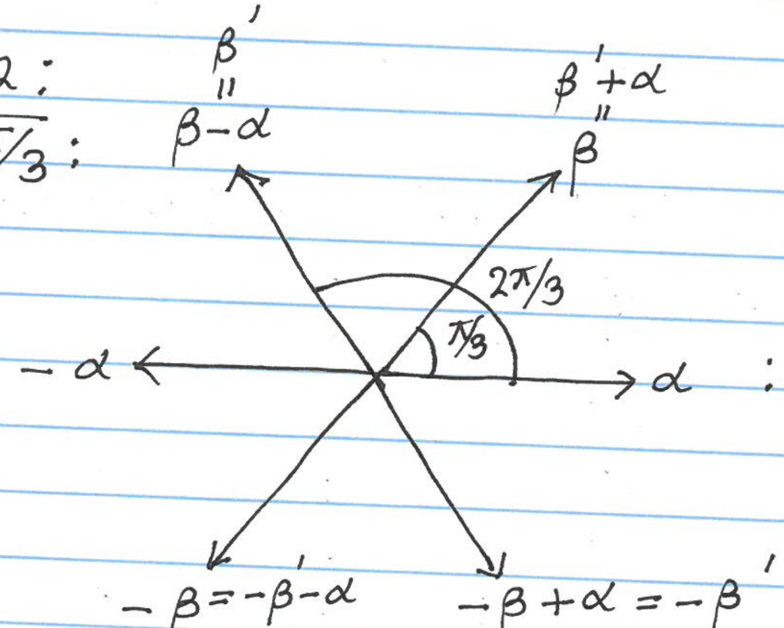
$$-\alpha \longleftarrow 0 \longrightarrow \alpha \; : \; \mathfrak{sl}(2,\mathbb{C})$$

Case 1:



$: \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$

Case 2:
$\theta = \pi/3$:



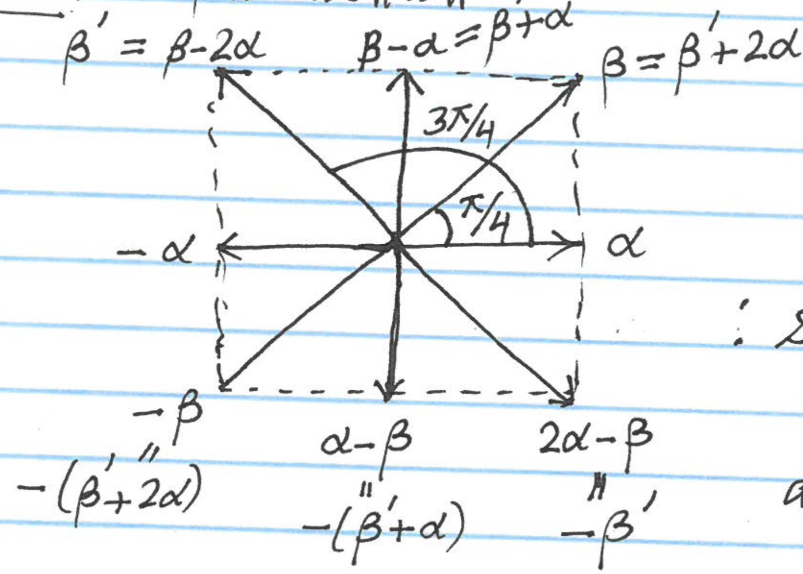
$\|\alpha\| = \|\beta\|$

$: \mathfrak{sl}(3,\mathbb{C}) = A_2$

$$\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha = \beta - \alpha$$

$$\Delta = \{\alpha, \beta', \beta'+\alpha, -\alpha, -\beta', -\beta'-\alpha\}$$

Case 3: $\|\beta\| = \sqrt{2}\,\|\alpha\|$,



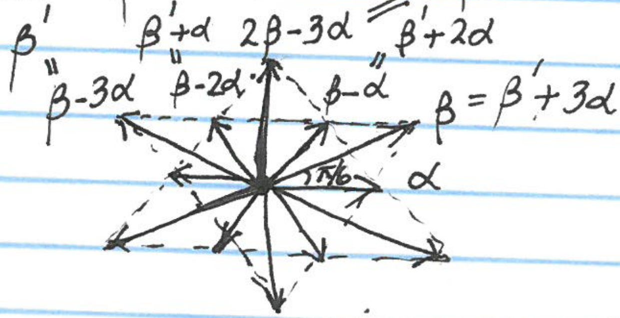
: $so(5,\mathbb{C}) = B_2$

$5 = 2l+1,\ l=2$

$\dim = 2l^2 + l = 10.$

$$\Delta = \{\alpha, \beta', \beta'+\alpha, \beta'+2\alpha, -\alpha, -\beta', -\beta'-\alpha, -\beta'-2\alpha\}$$

Case 4: $\|\beta\| = \sqrt{3}\,\|\alpha\|$ $\quad 2\beta'+3\alpha$



$$\Delta = \{\pm\alpha, \pm\beta', \pm\beta'+\alpha, \pm\beta'+2\alpha, \pm\beta'+3\alpha, \pm 2\beta'+3\alpha\}$$

Case for an exceptional 14-dim'l simple Lie algebra denoted by G_2.