L fin. dim'l semisimple Lie alg / $\mathbb{C}$
T max'l toral subalg.

$$L = T \oplus \bigoplus_{\alpha \in \Delta} L_\alpha .$$

$$E_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}}\{\Delta\}$$

$(\, , \,) : E_{\mathbb{Q}} \times E_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ is pos. def.

$$E = \mathbb{R} \otimes_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{\Delta\} \cong \operatorname{span}_{\mathbb{R}}\{\Delta\}.$$

Choose basis $\{\alpha_1, \alpha_2, \cdots, \alpha_\ell\} \subset \Delta$ for $E_{\mathbb{Q}}$

$\{1 \otimes \alpha_1, \cdots, 1 \otimes \alpha_\ell\}$ basis for E.

$\gamma, \delta \in E$

$$\gamma = \sum_{i=1}^{\ell} r_i \otimes \alpha_i \quad , \quad \delta = \sum_{i=1}^{\ell} s_i \otimes \alpha_i$$

Define

$$(\gamma, \delta) = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} r_i s_j (\alpha_i, \alpha_j) \in \mathbb{R}$$

Then $(\, , \,) : E \times E \longrightarrow \mathbb{R}$ is symm. nondeg. bil. form.

Prop: $(\, , \,) : E \times E \longrightarrow \mathbb{R}$ is pos. def.

Pf: $\gamma = \sum_{i=1}^{\ell} r_i \otimes \alpha_i \in E$

$$(\gamma,\gamma) = \sum_{i,j} r_i r_j (\alpha_i, \alpha_j)$$

$$= \sum_{i,j} r_i r_j \sum_{\beta\in\Delta} (\beta,\alpha_i)(\beta,\alpha_j)$$

$$= \sum_{\beta\in\Delta} \sum_{i,j} r_i r_j (\beta,\alpha_i)(\beta,\alpha_j)$$

$$= \sum_{\beta\in\Delta} \sum_{i,j} (1\otimes\beta, r_i\otimes\alpha_i)(1\otimes\beta, r_j\otimes\alpha_j)$$

$$= \sum_{\beta\in\Delta} \Big(1\otimes\beta, \underbrace{\sum_i (r_i\otimes\alpha_i)}_{=\gamma}\Big)\Big(1\otimes\beta, \underbrace{\sum_j (r_j\otimes\alpha_j)}_{=\gamma}\Big)$$

$$= \sum_{\beta\in\Delta} (1\otimes\beta, \gamma)^2 \geq 0 .$$

Suppose $(\gamma,\gamma)=0$. Then

$$\forall\ \beta\in\Delta,\quad (1\otimes\beta,\gamma)=0$$

$\Rightarrow (1\otimes\alpha_i, \gamma) = 0, \quad 1\leq i\leq l.$

$\Rightarrow (\delta,\gamma)=0 \quad \forall\ \delta\in E$

$\Rightarrow \gamma = 0$ since $(\ ,\ )$ is nondeg. on E. //

Hence $E = \mathrm{span}_{\mathbb{R}}\{\Delta\} \cong \mathbb{R}^{l}$.
where $l = \dim(E)$. (Note
$l = \dim E = \dim T^{*} = \dim T$)
l is called the rank of Δ.

Abstract Root System:

Let $E \cong \mathbb{R}^{l}$ be an l-dim'l Euclidean space with pos. def. symm, bil. form
$(\, , \,) : E \times E \longrightarrow \mathbb{R}$.

Defn. (Root system)

A nonempty subset $\Delta \subseteq E$ is called a root system if

(1) Δ is a finite set, $0 \notin \Delta$ and $E = \mathrm{span}_{\mathbb{R}}\{\Delta\}$.

(2) $\alpha \in \Delta$, $c\alpha \in \Delta \Rightarrow c = \pm 1$.

(3) $\alpha, \beta \in \Delta \Rightarrow \langle \alpha, \beta \rangle = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

(4) $\alpha, \beta \in \Delta \Rightarrow \beta - \langle \alpha, \beta \rangle \alpha \in \Delta$.

Prop: Let Δ be the set of roots of a fin. dim'l semisimple Lie alg. w.r.to a fixed max'l toral subalg. T. Then Δ is a root system.

Pf: (1), (2), (3) hold. Left to show that (4) holds. Let $\alpha, \beta \in \Delta$ and α-string through β be

$$\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha$$

where $r - q = \beta(h_\alpha) = \beta\left(\frac{2t_\alpha}{(\alpha,\alpha)}\right) = \frac{2\beta(t_\alpha)}{(\alpha,\alpha)}$

$$= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \langle \alpha, \beta \rangle$$

$\Rightarrow \beta - \langle \alpha,\beta\rangle\alpha = \beta - (r-q)\alpha = \beta + (q-r)\alpha$

and $-r \le q - r \le q$

$\Rightarrow \beta - \langle \alpha,\beta\rangle\alpha \in \Delta$ since the α-string through β is unbroken. //

Consider any root system Δ. $E = \mathrm{span}_{\mathbb{R}}\{\Delta\}$
For $\alpha \in \Delta$, define

$$\sigma_\alpha : E \longrightarrow E \text{ by } \sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha \ \forall \beta \in E$$

Since $\langle , \rangle$ is linear in the second coordinate σ_α is a lin. op. on E.

By (A), $\sigma_\alpha(\beta) \in \Delta \ \forall \beta \in \Delta$.

Since $(,)$ is pos. def. on E,

$$E = \mathbb{R}\alpha \oplus (\mathbb{R}\alpha)^{\perp} = \mathbb{R}\alpha \oplus H_\alpha, \quad H_\alpha = (\mathbb{R}\alpha)^{\perp}$$

Let $\gamma \in H_\alpha \Rightarrow (\gamma, \alpha) = 0$

$$\Rightarrow \sigma_\alpha(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha = \gamma - \frac{2(\alpha,\gamma)}{(\alpha,\alpha)}\alpha = \gamma$$

$$\Rightarrow \sigma_\alpha \Big|_{H_\alpha} = id$$

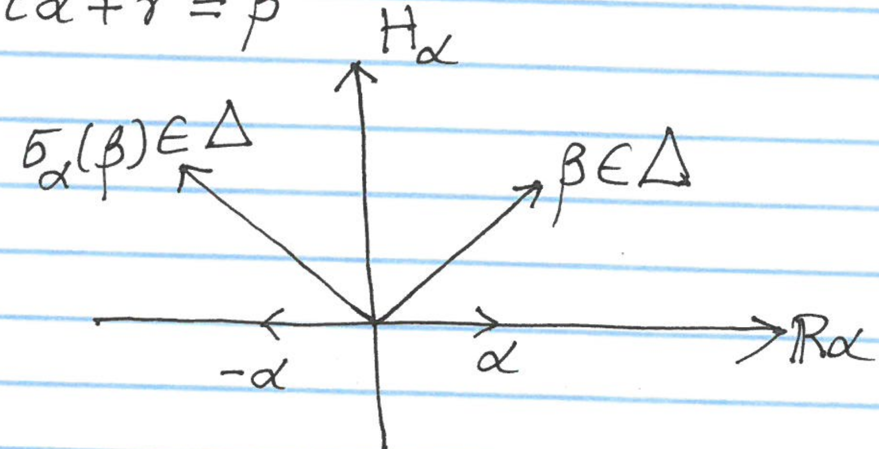
We call H_α the α-hyperplane.

Let $\beta \in E = \mathbb{R}\alpha \oplus H_\alpha \Rightarrow \beta = c\alpha + \gamma$, $\gamma \in H_\alpha$, $c \in \mathbb{R}$

$$\sigma_\alpha(\beta) = c\,\sigma_\alpha(\alpha) + \gamma = -c\alpha + \gamma$$

since $\sigma_\alpha(\alpha) = -\alpha$.

$$\Rightarrow \sigma_\alpha^2(\beta) = \sigma_\alpha(-c\alpha+\gamma) = -c\,\sigma_\alpha(\alpha)+\gamma$$
$$= c\alpha+\gamma = \beta$$
$$\Rightarrow \sigma_\alpha^2 = id$$



$\forall\, \alpha \in \Delta$, σ_α is called a reflection on H_α.

W = subgroup of $Aut(E)$ gen. by $\{\sigma_\alpha \mid \alpha \in \Delta\}$.

called the Weyl group associated with Δ.

$\sigma \in W \Rightarrow \sigma = \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_k}$, $\alpha_1, \dots, \alpha_k \in \Delta$.

<u>Prop:</u> W is a finite group.
<u>Pf:</u> Δ is a finite set containing a basis for E. By axiom (4), $\sigma_\alpha : \Delta \to \Delta$ $\forall \alpha \in \Delta$.
Let $\psi \in W$. Then $\psi = \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_k}$, $\alpha_1, \dots, \alpha_k \in \Delta$
Let $|\Delta| = n$. Then $\sigma_{\alpha_i} \in S_n$. Hence $\psi \in S_n$.
$\Rightarrow W \subseteq S_n \Rightarrow W$ finite.