L fin. dim'l s.s. Lie alg / $\mathbb{C}$
T max'l toral subalg of L

$$L = T \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$$

$$T^* = \text{span}\{\Delta\}$$

Choose a basis $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\} \subset \Delta$ for T^*.
$\forall\, \beta \in T^*$,

$$\beta = \sum_{i=1}^{\ell} c_i \alpha_i \quad , \quad c_i \in \mathbb{C}$$

<u>Prop</u>: $\forall\, \beta \in T^*, \beta \in \Delta, \beta = \sum_{i=1}^{\ell} c_i \alpha_i \ , \ c_i \in \mathbb{Q}$.

<u>Pf</u>: Recall

$$\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \left(\frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \alpha_j \right)$$

$$(\alpha_i, \alpha_j) = \kappa(t_{\alpha_i}, t_{\alpha_j})$$

$$\left(\frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \alpha_j \right) = \kappa\left(\frac{2t_{\alpha_i}}{(\alpha_i, \alpha_i)}, t_{\alpha_j} \right)$$

$$= \alpha_j\left(\frac{2t_{\alpha_i}}{(\alpha_i, \alpha_i)} \right) = \alpha_j(h_{\alpha_i})$$

$$\Rightarrow \langle \alpha_i, \alpha_j \rangle = \alpha_j(h_{\alpha_i}).$$

Consider $A = \left(\langle \alpha_i, \alpha_j \rangle \right)_{\ell \times \ell}$

Since $(\ ,\ ): T^* \times T^* \longrightarrow \mathbb{C}$ is nondeg.

$((\alpha_i, \alpha_j))_{l \times l}$ is nonsingular.

$\Rightarrow \quad A = (\langle \alpha_i, \alpha_j \rangle)_{l \times l}$ is nonsingular.

Take $\beta \in T^*$, $\beta = \sum_{i=1}^{l} c_i \alpha_i$

$$(\beta, \alpha_j) = \sum_{i=1}^{l} c_i (\alpha_i, \alpha_j)$$

$$\shortparallel$$

$$(\alpha_j, \beta) = \sum_{i=1}^{l} c_i (\alpha_j, \alpha_i)$$

$$\langle \alpha_j, \beta \rangle = \left\langle \frac{2\alpha_j}{(\alpha_j, \alpha_j)}, \beta \right\rangle$$

$$= \frac{2}{(\alpha_j, \alpha_j)} (\alpha_j, \beta)$$

$$= \frac{2}{(\alpha_j, \alpha_j)} \sum_{i=1}^{l} c_i (\alpha_j, \alpha_i)$$

$$= \sum_{i=1}^{l} c_i \langle \alpha_j, \alpha_i \rangle, \quad 1 \le j \le l.$$

which is $(l \times l)$-lin. system with coefficient matrix

$$A = (\langle \alpha_j, \alpha_i \rangle)_{l \times l} \in \mathbb{Z}^{l \times l}$$

$\Rightarrow A^{-1} \in \mathbb{Q}^{l \times l} \Rightarrow c_i \in \mathbb{Q}$. //

<u>Prop</u>: $\forall\ \gamma, \delta \in T^*$

$$(\gamma, \delta) = \sum_{\beta \in \Delta} (\beta, \gamma)(\beta, \delta)$$

<u>Pf</u>:

$$(\gamma, \delta) = \kappa(t_\gamma, t_\delta) = \mathrm{Tr}\left(ad_{t_\gamma}\, ad_{t_\delta}\right)$$

Observe $ad_{t_\gamma}\, ad_{t_\delta} = ad_{t_\delta}\, ad_{t_\gamma}$ since $[t_\gamma, t_\delta] = 0$ (T abelian)

Also $ad_{t_\gamma}, ad_{t_\delta}$ are semisimple since T is toral.

So we can choose an order basis $\mathcal{B}$ for L such that

$$[ad_{t_\gamma}]_{\mathcal{B}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \beta(t_\gamma) & & \\ & & & & \alpha(t_\gamma) & \\ & & & & & \ddots \end{pmatrix}$$

and $$[ad_{t_\delta}]_{\mathcal{B}} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \beta(t_\delta) & & \\ & & & & \alpha(t_\delta) & \\ & & & & & \ddots \end{pmatrix}$$

where $\beta, \alpha, \ldots \in \Delta$.

$$(\gamma, \delta) = \mathrm{Tr}\left(ad_{t_\gamma}\, ad_{t_\delta}\right) = \sum_{\beta \in \Delta} \overset{(\beta,\gamma)}{\overset{\|}{\beta(t_\gamma)}}\, \overset{(\beta,\delta)}{\overset{\|}{\beta(t_\delta)}}.$$

$$= \sum_{\beta \in \Delta} (\beta, \gamma)(\beta, \delta) \quad //$$

Cor: $\forall\, \alpha \in \Delta$, $(\alpha, \alpha) \in \mathbb{Q}^+ = \mathbb{Q}_{>0}$

Pf: Know $(\alpha,\alpha) = \kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0$.

$$(\alpha,\alpha) = \sum_{\beta\in\Delta} (\beta,\alpha)(\beta,\alpha) = \sum_{\beta\in\Delta} (\beta,\alpha)^2.$$

Recall $\forall\, \alpha, \beta \in \Delta$ $\langle \alpha, \beta\rangle \in \mathbb{Z}$

$$\langle \alpha,\beta\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \beta(h_\alpha) = r_\beta - q_\beta \in \mathbb{Z}$$

where $\beta - r_\beta\alpha, \ldots, \beta, \ldots, \beta + q_\beta \alpha$

is the α-string through β.

$$\Rightarrow \underset{\substack{\shortparallel \\ (\beta,\alpha)}}{(\alpha,\beta)} = \left(\frac{r_\beta - q_\beta}{2}\right)(\alpha,\alpha)$$

$$\Rightarrow (\beta,\alpha)^2 = \frac{(r_\beta - q_\beta)^2}{4}(\alpha,\alpha)^2$$

$$\Rightarrow (\alpha,\alpha) = \sum_{\beta\in\Delta} \frac{(r_\beta - q_\beta)^2}{4}(\alpha,\alpha)^2$$

$$\Rightarrow \frac{1}{(\alpha,\alpha)} = \sum_{\beta\in\Delta} \frac{(r_\beta - q_\beta)^2}{4} \in \mathbb{Q}_{>0}$$

$$\Rightarrow (\alpha,\alpha) \in \mathbb{Q}_{>0} \quad //.$$

In summary, we have

$\forall\ \alpha \in \Delta$, $(\alpha, \alpha) \in \mathbb{Q}_{>0}$

$\forall\ \alpha, \beta \in \Delta$, $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

$\Rightarrow (\alpha, \beta) = (\beta, \alpha) \in \mathbb{Q}$.

Set $E_{\mathbb{Q}} = \text{span}_{\mathbb{Q}}\{\Delta\}$

Then $(\ ,\ ) : E_{\mathbb{Q}} \times E_{\mathbb{Q}} \to \mathbb{Q}$

is a nondeg. symm. bil. form.

<u>Prop</u>: $(\ ,\ ) : E_{\mathbb{Q}} \times E_{\mathbb{Q}} \to \mathbb{Q}$ is positive definite.

<u>Pf</u>: Let $\gamma \in E_{\mathbb{Q}}$

Need to show: $(\gamma, \gamma) \geq 0$ &
$(\gamma, \gamma) = 0 \Leftrightarrow \gamma = 0$.

$$(\gamma, \gamma) = \sum_{\beta \in \Delta} (\beta, \gamma)(\beta, \gamma) = \sum_{\beta \in \Delta} (\beta, \gamma)^2 \in \mathbb{Q}_{\geq 0}$$

since $\gamma \in E_{\mathbb{Q}}$ & $(\beta, \alpha) \in \mathbb{Q}$ $\forall \alpha \in \Delta$.

$\Rightarrow (\gamma,\gamma) \geq 0.$

Enough to show $(\gamma,\gamma)=0 \Rightarrow \gamma=0.$

$(\gamma,\gamma)=0 \Rightarrow (\beta,\gamma)=0 \;\; \forall \beta \in \Delta$

Since $T^* = \mathrm{span}_{\mathbb{C}}\{\Delta\}$ we have

~~$\forall \beta \in T^*, (\beta,\gamma)$~~

$\forall \; \delta \in T^*, \; (\delta,\gamma)=0$ since

$(\beta,\gamma)=0 \;\; \forall \beta \in \Delta.$

$\Rightarrow \gamma \in \mathrm{rad}\,(\,,\,)\big|_{T^* \times T^*} = 0$

since $(\,,\,)\big|_{T^* \times T^*}$ is nondeg.

$\Rightarrow \gamma = 0 \Rightarrow (\,,\,): E_{\mathbb{Q}} \times E_{\mathbb{Q}} \longrightarrow \mathbb{Q}$

is positive definite. //

Set $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}} = \mathrm{span}_{\mathbb{R}}\{\Delta\}.$

$(r \otimes \beta) \longmapsto r\beta$

Fix basis $\{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ for $E_{\mathbb{Q}}$

$\Rightarrow \{1 \otimes \alpha_1, 1 \otimes \alpha_2, \cdots, 1 \otimes \alpha_\ell\}$ basis for E

Define $(\ ,\ ): E \times E \to \mathbb{R}$ ~~by~~ as follows.

$\cancel{(\gamma, \delta)}$

For $\gamma, \delta \in E$,

$$\gamma = \sum_{i=1}^{\ell} \underbrace{r_i \otimes \alpha_i}_{= r_i(1 \otimes \alpha_i)}, \quad \delta = \sum_{i=1}^{\ell} s_i \otimes \alpha_i$$

Define $(\ ,\ ): E \times E \to \mathbb{R}$ by

$$(\gamma, \delta) = \sum_{1 \le i \le \ell} \sum_{1 \le j \le \ell} r_i s_j (\alpha_i, \alpha_j) \in \mathbb{R}$$

Since $(\ ,\ ): E_{\mathbb{Q}} \times E_{\mathbb{Q}} \to \mathbb{Q}$ is nondeg. symm. bil. form, we have $(\ ,\ ): E \times E \to \mathbb{R}$ is nondeg. symm. bil. form.

<u>Prop:</u> $(\ ,\ ): E \times E \to \mathbb{R}$ is pos. def.

($\Rightarrow E \cong \mathbb{R}^{\ell}$, Euclidean space.)