Cor: $\alpha, \beta \in \Delta$, $\alpha+\beta \in \Delta$. Then

$$[L_\alpha, L_\beta] = L_{\alpha+\beta}$$

Pf: Know $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ &

$\dim L_{\alpha+\beta} = 1$.

Enough to show that $[L_\alpha, L_\beta] \neq 0$.

Recall $M = \sum_{i \in \mathbb{Z}} L_{\beta+i\alpha}$

is an irred. S_α-module. By $sl(2)$-Theory

$M \cong V(m)$ for some $m \in \mathbb{Z}_{\geq 0}$.

$$\cong L_{\beta+q\alpha} + \cdots + L_{\beta+\alpha} + L_\beta + \cdots + L_{\beta-r\alpha}$$

$\dim L_{\beta+i\alpha} = 1$, $-r \leq i \leq q$.

Suppose
$$\begin{aligned}
L_{\beta+q\alpha} &= \operatorname{span}\{v_0\} \\
L_{\beta+(q-1)\alpha} &= \operatorname{span}\{v_1\} \\
&\vdots \\
L_{\beta+\alpha} &= \operatorname{span}\{v_{q-1}\} \\
L_\beta &= \operatorname{span}\{v_q\} \\
&\vdots
\end{aligned}$$

By $sl(2)$-theory

$$[\underset{\substack{\cap \\ L_{\beta+\alpha}}}{e_\alpha, v_q}] = (m-q+1)\, v_{q-1}$$

$$m = \beta(h_\alpha) + 2q = (r-q) + 2q = r+q$$

$$\Rightarrow m-q+1 = r+q-q+1 = r+1 \neq 0$$

$$\Rightarrow [e_\alpha, v_q] \neq 0 \Rightarrow [L_\alpha, L_\beta] \neq 0$$

$$\Rightarrow [L_\alpha, L_\beta] = L_{\alpha+\beta}. \qquad //$$

Cor: $\alpha \in \Delta$, $c\alpha \in \Delta \Rightarrow c = \pm 1$.

Pf: If $c \in \mathbb{Z}$, then statement holds.

Assume $c \notin \mathbb{Z}$.

Set $\beta = c\alpha \in \Delta$

Know $\beta(h_\alpha) = r - q \in \mathbb{Z}$

$$\begin{array}{c} \beta(h_\alpha) \\ \| \\ c\alpha(h_\alpha) = 2c \end{array}$$

$$\Rightarrow 2c \in 2\mathbb{Z}+1$$

As before $M = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$ is an irreducible S_α-module.

and $M = L_{\beta+q\alpha} + \cdots + L_{\beta} + \cdots + L_{\beta - r\alpha}$

Eigenvalues of h_α on M are

$$\beta(h_\alpha)+2q$$
$$\vdots$$
$$\beta(h_\alpha) = 2c \in 2\mathbb{Z}+1$$
$$\vdots$$
$$\beta(h_\alpha)-2r$$

$\Rightarrow$ 1 and -1 occur as eigenvalues of h_α on M.

Suppose $(\beta+i\alpha)(h_\alpha) = \beta(h_\alpha)+2i = 1$

$-r \leq i \leq q$

$\Rightarrow 1 = \beta(h_\alpha)+2i = c\alpha(h_\alpha)+2i$

$= 2c+2i = 2(c+i) = 2k$, $k = c+i$

$\Rightarrow \beta+i\alpha = c\alpha+i\alpha = (c+i)\alpha = k\alpha$

$\Rightarrow (\beta+i\alpha)(h_\alpha) = k\alpha(h_\alpha) = 2k = 1$

$\Rightarrow k = \frac{1}{2}$ and $\frac{1}{2}\alpha = \beta+i\alpha \in \Delta$

But $2\left(\frac{1}{2}\alpha\right) = \alpha \in \Delta$ which is a contradiction. Hence $c \notin \mathbb{Z} \Rightarrow c\alpha \notin \Delta$. //

Prop: $\Delta = \{\alpha, -\alpha\}$ are the set of roots of T on L. Then $L \cong sl(2,\mathbb{C})$.

Pf: Know $L = \underset{\substack{\| \\ (\ker\alpha) + \mathbb{C}h_\alpha}}{T} + L_\alpha + L_{-\alpha}$

$$\Rightarrow L = (\ker\alpha) + \underbrace{\mathbb{C}h_\alpha + \mathbb{C}e_\alpha + \mathbb{C}f_{\alpha}}_{\substack{\cong \\ sl(2,\mathbb{C})}}$$

Enough to show $\ker\alpha = 0$.

Let $t \in \ker\alpha \subseteq T$

$[t, T] = 0$ since T abelian

$[t, e_\alpha] = \alpha(t)e_\alpha = 0$ since $t \in \ker\alpha$

$[t, f_\alpha] = -\alpha(t)f_\alpha = 0$

$\Rightarrow [t, L] = 0 \Rightarrow t \in Z(L) = 0 \Rightarrow t = 0$

$\Rightarrow \ker\alpha = 0$

$\Rightarrow L = \mathbb{C}h_\alpha + \mathbb{C}e_\alpha + \mathbb{C}f_\alpha \cong sl(2,\mathbb{C})$. //

L fin. dim'l semisimple Lie alg/$\mathbb{C}$

T max'l toral subalg.

$$L = T \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$$

where Δ is the set of roots of T on L, & $\dim L_\alpha = 1$ $\forall\, \alpha \in \Delta$.

$\kappa(\ ,\ )$, Killing form is nondeg. on L

Also $\kappa(\ ,\ )\big|_{T \times T}$ is nondeg.

For $\alpha \in \Delta$ $\exists!\ t_\alpha \in T$ such that

$\forall\ t \in T$, $\kappa(t, t_\alpha) = \alpha(t)$

In particular, $\kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0$.

$\exists$ $e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$ s.t.

$\operatorname{span}\{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\} \cong sl(2, \mathbb{C})$

where $h_\alpha = \dfrac{2t_\alpha}{\alpha(t_\alpha)}$

<u>Note:</u> $\Delta \subset T^* \cong T$ (as vector space)

<u>Prop</u>: $\operatorname{span}_{\mathbb{C}}\{\Delta\} = T^*$

<u>Pf</u>: $\operatorname{span}_{\mathbb{C}}\{\Delta\} = S^* \subset T^*$

Choose basis $\{\alpha_1, \alpha_2, \cdots, \alpha_s\} \subset \Delta$ for S^*

Extend it to a basis

$$\{\alpha_1, \cdots, \alpha_s, \alpha_{s+1}, \cdots, \alpha_n\} \text{ for } T^*.$$

Suppose $s < n$. Choose dual basis

$\{t_1, t_2, \cdots, t_n\}$ for T such that

$$\alpha_j(t_i) = \delta_{ij}$$

$\Rightarrow \alpha_j(t_n) = 0, \quad j = 1, 2, \cdots, s.$

$\Rightarrow \forall\, \beta \in S^*, \ \beta(t_n) = 0.$

$\Rightarrow \forall\, \beta \in \Delta, \ \beta(t_n) = 0$

$\Rightarrow [t_n, L_\beta] = 0 \quad \forall\, \beta \in \Delta$

& $[t_n, T] = 0$ since T abelian.

$\Rightarrow [t_n, L] = 0 \Rightarrow t_n \in Z(L) = 0$

$\Rightarrow t_n = 0$ which is a contradiction.

$\Rightarrow \quad s = n \Rightarrow T^* = S^* = \mathrm{span}_{\mathbb{C}}\{\Delta\}$. //

Now choose a basis

$$\{\alpha_1, \alpha_2, \cdots, \alpha_\ell\} \subset \Delta$$

for T^*. Then

$$\forall \beta \in T^*, \quad \beta = \sum_{i=1}^{\ell} c_i \alpha_i, \quad c_i \in \mathbb{C}.$$

For $\alpha_i \in \Delta$, $\exists!\, t_{\alpha_i} \in T$ s.t.

$$\kappa(t, t_{\alpha_i}) = \alpha_i(t) \quad \forall t \in T.$$

$\Rightarrow \forall \beta \in T^* \; \exists!\, t_\beta \in T$ such that

$$\kappa(t, t_\beta) = \beta(t) \quad \forall t \in T.$$

So for $\alpha, \beta \in T^*$ $\exists!\, t_\alpha, t_\beta \in T$ s.t. $\forall t \in T$

$$\kappa(t, t_\alpha) = \alpha(t), \quad \kappa(t, t_\beta) = \beta(t)$$

Define $(\ ,\ ): T^* \times T^* \longrightarrow \mathbb{C}$

by $\quad (\alpha, \beta) = \kappa(t_\alpha, t_\beta) \quad \forall \alpha, \beta \in T^*$

Since $\varkappa(\ ,\ ): T \times T \to \mathbb{C}$ is a symm. nondeg. bilinear form, we have $(\ ,\ ): T^* \times T^* \to \mathbb{C}$ is a symm., nondeg. bilinear form.

Now we have:

(1) $\forall\ \alpha \in \Delta \subset T^*$

$$(\alpha, \alpha) = \varkappa(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0.$$

(2) $\forall\ \alpha, \beta \in \Delta$

$$\beta(h_\alpha) = r - q \in \mathbb{Z}$$

where $\beta - r\alpha, \cdots, \beta, \cdots, \beta + q\alpha$ is the α-root string through β.

$$r - q = \beta(h_\alpha) = \beta\left(\frac{2t_\alpha}{\alpha(t_\alpha)}\right) = \frac{2\beta(t_\alpha)}{(\alpha, \alpha)}$$

$$= \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} := \langle \alpha, \beta \rangle$$

$$\Rightarrow \alpha, \beta \in \Delta,\ \langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

(Note: $\langle\ ,\ \rangle$ is linear in the second coordinate.)