$L$ fin. dim'l semisimple Lie alg. $/\mathbb{C}$

$T \subseteq L$ maximal toral subalg.

$\Delta$ = set of roots of $T$ on $T$.

$$L = T \oplus \bigoplus_{\alpha \in \Delta} L_\alpha .$$

$\alpha \in \Delta \Rightarrow \exists \ \{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\} \in L$

s.t. $\operatorname{span}\{e_\alpha, f_\alpha, h_\alpha\} \cong sl(2, \mathbb{C})$.

$\alpha \in \Delta \Rightarrow -\alpha \in \Delta$.

Thm: (1) $\alpha \in \Delta \Rightarrow \dim L_\alpha = 1$

(2) $\alpha \in \Delta$, $k \in \mathbb{Z}$,

$k\alpha \in \Delta \iff k = \pm 1$.

Pf: Fix $\alpha \in \Delta$.

Choose $0 \neq e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$, $h_\alpha \in T$ such that $S_\alpha = \operatorname{span}\{e_\alpha, f_\alpha, h_\alpha\} \cong sl(2, \mathbb{C})$

Recall $\alpha(h_\alpha) = 2$.

Consider $$M = \sum_{j \in \mathbb{Z}} L_{j\alpha}$$

Recall $T = L_0 \Rightarrow T \subseteq M$

$\alpha : T \longrightarrow \mathbb{C}$ lin. functional

$\alpha(h_\alpha) = 2 \neq 0 \Rightarrow \alpha$ is onto.

$\Rightarrow \dim(\ker \alpha) = (\dim T) - 1$

Recall

$$[e_\alpha, L_{j\alpha}] \subseteq L_{(j+1)\alpha}$$

$$[f_\alpha, L_{j\alpha}] \subseteq L_{(j-1)\alpha}$$

$$[h_\alpha, L_{j\alpha}] \subseteq L_{j\alpha}$$

$\Rightarrow M$ is a $S_\alpha$-module under adjoint action.

$M \subseteq L \Rightarrow M$ fin. dim'l.

$\Rightarrow$ By $sl(2, \mathbb{C})$ repn. Theory that

$$M = V(m_1) \oplus V(m_2) \oplus \cdots \oplus V(m_r)$$

for some $m_1, m_2, \cdots, m_r \in \mathbb{Z}_{\geq 0}$ where $V(m_i)$ are irred. $S_\alpha$-submodules.

The eigenvalues of $h_\alpha$ on $M$ are

$$j\alpha(h_\alpha) = 2j$$

$\Rightarrow m_i = 2n_i, \quad 1 \leq i \leq r.$

Recall

$$r = \dim M_0 + \dim M_1$$

Since $M_1 = \{0\}$, we have $r = \dim M_0$.

Recall that in $V(m_i) = V(2n_i)$, the eigenvalues of $h_\alpha$ are:

$$2n_i,\ 2n_i - 2,\ \dots,\ 0,\ \dots,\ -2n_i + 2,\ -2n_i$$

with each eigenspace being 1 dim'l.

$\Rightarrow$ We get the 0-eigenvalue of $h_\alpha$ only on $\underset{\substack{\parallel \\ M_0}}{L_0} = T$.

$\Rightarrow \quad r = \dim(T)$

Since $\alpha(h_\alpha) = 2$, $h_\alpha \notin \ker\alpha \subseteq T$

$\Rightarrow \dim(T) = \dim(\ker\alpha) + 1$

$\Rightarrow \quad T = (\ker\alpha) \oplus \mathbb{C} h_\alpha$

For $t \in \ker(\alpha) \Rightarrow \alpha(t) = 0$

$\Rightarrow \quad [t, e_\alpha] = \alpha(t) e_\alpha = 0$

$[t, f_\alpha] = -\alpha(t) f_\alpha = 0$

$[t, h_\alpha] = 0$ since $T$ is abelian.

$\Rightarrow$ $\forall\ t \in \ker(\alpha)$

$\mathbb{C}t \cong V(0)$ irred. $S_\alpha$-submodule.

$\Rightarrow$ $\ker\alpha \cong \underbrace{V(0) \oplus \cdots \oplus V(0)}_{(\dim T)-1}$

However, $S_\alpha = \operatorname{span}\{e_\alpha, f_\alpha, h_\alpha\}$

$= \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha \oplus \mathbb{C}h_\alpha \subseteq M$

$\Rightarrow$

$\ker\alpha + S_\alpha$ is a $S_\alpha$-submodule

$\ker\alpha + S_\alpha = \underbrace{V(0) \oplus \cdots \oplus V(0)}_{\dim T - 1} + S_\alpha$

$= \underbrace{\underbrace{V(0) \oplus \cdots \oplus V(0)}_{\dim T - 1} \oplus \mathbb{C}h_\alpha} \oplus \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha$

$= T \oplus \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha$

The 0-eigenspace of $h_\alpha$ on $\ker\alpha \oplus S_\alpha$ has dimension $\dim T = \dim L_0 = r$

$$\Rightarrow \underset{\substack{\| \\ \sum_{j\in\mathbb{Z}} L_{j\alpha}}}{M} = \underset{\substack{\| \\ L_0}}{T} \oplus \underset{\substack{\wr \\ \cap \\ L_\alpha}}{\mathbb{C}e_\alpha} \oplus \underset{\substack{\wr \\ \cap \\ L_{-\alpha}}}{\mathbb{C}f_\alpha}$$

$\Rightarrow L_\alpha = \mathbb{C} e_\alpha, \ L_{-\alpha} = \mathbb{C} f_\alpha$

$\Rightarrow \dim L_\alpha = 1$ & $k\alpha \in \Delta \Leftrightarrow k = \pm 1$. //

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$\alpha, \beta \in \Delta, \quad \beta \neq \pm\alpha$

$\beta + i\alpha \in \Delta$ for which $i \in \mathbb{Z}$?

Thm: $\alpha, \beta \in \Delta, \quad \beta \neq \pm\alpha$

Suppose $r, q$ be the largest nonnegative integer such that

$$\beta - r\alpha, \ \beta + q\alpha \in \Delta$$

Then

(1) $\beta + i\alpha \in \Delta \ \forall -r \le i \le q$

(called the $\beta$-root string through $\alpha$)

(2) $r - q = \beta(h_\alpha)$ (Recall $h_\alpha = \frac{2}{\alpha(t_\alpha)} t_\alpha$)

Pf: Set

$$M = \sum_{j \in \mathbb{Z}} L_{\beta + j\alpha} \subseteq L$$

As before, $M$ is a fin. dim'l $S_\alpha$-module under adjoint action.

Eigenvalues of $h_\alpha$ on M:

$$(\beta+q\alpha)(h_\alpha) = \beta(h_\alpha)+2q$$

$$\vdots$$

$$\beta(h_\alpha)$$

$$\vdots$$

$$(\beta-r\alpha)(h_\alpha) = \beta(h_\alpha)-2r$$

By previous Thm. $\dim L_{\beta+j\alpha} \leq 1 \; \forall j$

Hence each eigenvalue of $h_\alpha$ on M appears once.

Since 0 and 1 can not occur as eigenvalues together &

$$M = W_1 \oplus \cdots \oplus W_r, \; W_i \text{ irred. } S_\alpha\text{-modules}$$

where $r = \dim M_0 + \dim M_1 = 1$

$\Rightarrow$ M is an irred. $S_\alpha$-module and

$$M \cong V(m) \quad \text{for some } m \in \mathbb{Z}_{\geq 0}$$

$\Rightarrow$ the eigenvalues of $h_\alpha$ on M are

$$m, m-2, \cdots, -m+2, -m$$

$$\Rightarrow \quad (\beta + q\alpha)(h_\alpha) = \beta(h_\alpha) + 2q = m$$

$$\& \quad (\beta - r\alpha)(h_\alpha) = \beta(h_\alpha) - 2r = -m$$

$$\Rightarrow \quad \beta(h_\alpha) + 2q = -\beta(h_\alpha) + 2r$$

$$\Rightarrow \quad 2r - 2q = 2\beta(h_\alpha) \Rightarrow r - q = \beta(h_\alpha)$$

Furthermore, $\beta(h_\alpha) + 2i$, $-r \leq i \leq q$ occur as an eigenvalue of $h_\alpha$ on $M$.

$$\Rightarrow \underset{r-q}{\underset{\parallel}{\beta(h_\alpha)}} \in \mathbb{Z} \quad \& \quad \beta + i\alpha \in \Delta \ \forall -r \leq i \leq q. //$$

Cor: $\alpha, \beta \in \Delta$ & $\beta + \alpha \in \Delta$

$$\Longrightarrow \quad [L_\alpha, L_\beta] = L_{\alpha+\beta}$$

Pf: