~~Rmk~~. L semisimple Lie alg. $/\mathbb{C}$.
T toral subalgebra of L.

$$L = L_0 \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$$

Thm: T max'l toral subalg. $\Leftrightarrow T = L_0$

Pf: ($\Longrightarrow$) T toral $\Rightarrow T$ abelian $\Rightarrow T \subseteq L_0$.

Let $0 \neq x \in L_0$. Then $x = t_x + x_\eta$ where $t_x \in T$, $x_\eta \in L_0$, $[t_x, x_\eta] = 0$ & ad_{x_η} nilp.

Since L_0 abelian, $\forall\, y \in L_0$

$$[ad_y, ad_{x_\eta}] = ad_{[y, x_\eta]} = 0$$

Since ad_{x_η} nilp. we have $ad_y\, ad_{x_\eta}$ is nilpotent.

$\Rightarrow \kappa(y, x_\eta) = tr(ad_y ad_{x_\eta}) = 0 \ \forall\, y \in L_0$

$\Longrightarrow x_\eta \in rad\, \kappa(\,,\,)\big|_{L_0} = 0$, since $\kappa(\,,\,)\big|_{L_0}$ is nondeg.

$\Longrightarrow x = t_x \in T \Rightarrow L_0 \subseteq T \Rightarrow T = L_0$.

($\Leftarrow$) Suppose $T = L_0$

Suppose T' is a toral subalg. & $T' \supseteq T$.
T' toral $\Rightarrow$ T' abelian
$\Rightarrow T' \subseteq L_0 = T \Rightarrow T' = T$ & T is max'l. //

From now on fix T to be a max'l toral subalg. Then

$$L = T \oplus \bigoplus_{\alpha \in \Delta} L_\alpha .$$

<u>Remark</u>: Maximal toral subalg. are not unique.

<u>Ex (1)</u> $L = sl(2, \mathbb{C}) = \mathrm{span}\{e, f, h\}$

$T = \mathrm{span}\{h\}$ is a max'l toral subalg.

Recall that ad_{e+f} is semisimple.

$T' = \mathrm{span}\{e+f\}$ is also a max'l toral subalg.

Ex(2) $L = sl(n, \mathbb{C})$

$$E = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \quad \text{Coxeter element.}$$

$T = \text{span}\{E, E^2, \cdots, E^{n-1}\}$ is a max'l toral subalg.

Of course, $\text{span}\{E_{ii} - E_{i+1,i+1} \mid 1 \le i \le n-1\}$ is a max'l toral subalg.

Lemma: $\forall\ \alpha, \beta \in \Delta \cup \{0\}$

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}.$$

Pf: $x \in L_\alpha$, $y \in L_\beta$, $t \in T$

$$\begin{aligned}[t,[x,y]] &= [[t,x],y] + [x,[t,y]] \\ &= \alpha(t)[x,y] + \beta(t)[x,y] \\ &= (\alpha+\beta)(t)\,[x,y]. \quad //\end{aligned}$$

Prop: For $\beta \in T^*$, $\exists!\ t_\beta \in T$ such that $\beta(t) = \kappa(t, t_\beta)\ \forall\ t \in T$.

<u>Pf:</u> Recall $\kappa(\,,\,)|_T$ nondeg.

For $s \in T$ define
$$\kappa(s,-) : T \longrightarrow \mathbb{C}$$
by $\kappa(s,-)(t) = \kappa(s,t) \;\forall\, t \in T$

Then $\kappa(s,-) \in T^*$ since $\kappa(\,,\,)$ bilinear.

Define $\varphi : T \longrightarrow T^*$ by
$$\varphi(s) \longmapsto \kappa(s,-) \quad \forall\, s \in T.$$

Clearly φ is linear. Let $s \in \ker\varphi$

$\Rightarrow \varphi(s) = 0 \Rightarrow \kappa(s,-) = 0$

$\Rightarrow \kappa(s,t) = \kappa(s,-)(t) = 0 \quad \forall\, t \in T$

$\Rightarrow s = 0$ since $\kappa(\,,\,)|_T$ nondeg.

$\Rightarrow \varphi$ is an isom. of vector spaces.

$\Rightarrow \forall\, \beta \in T^* \;\exists!\; t_\beta \in T$ such that
$$\varphi(t_\beta) = \beta$$

$$\Rightarrow \underbrace{\kappa(t_\beta,-)(t)}_{\kappa(t_\beta,t) \,=\, \kappa(t,t_\beta)} = \beta(t) \quad \forall\, t \in T$$

//

Thm: $\forall\ \alpha \in \Delta \subseteq T^*$ choose $t_\alpha \in T$ such that $\alpha(t) = \kappa(t, t_\alpha)$ $\forall\ t \in T$. Then

(1) $\forall\ e \in L_\alpha$, $f \in L_{-\alpha}$, we have

$$[e, f] = \kappa(e, f)\, t_\alpha$$

$$(\Rightarrow [L_\alpha, L_{-\alpha}] = \mathbb{C} t_\alpha)$$

(2) $\alpha(t_\alpha) \neq 0$

(3) For $0 \neq e_\alpha \in L_\alpha$, then $\exists\ 0 \neq f_\alpha \in L_{-\alpha}$ such that $\operatorname{span}\{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\} \cong sl(2, \mathbb{C})$.

(4) $h_\alpha = \dfrac{2}{\alpha(t_\alpha)} t_\alpha$ $(\Rightarrow \alpha(h_\alpha) = 2)$.

Pf (1) Let $t \in T$

$$\kappa(t, [e, f]) = \kappa([t, e], f) = \kappa(\alpha(t) e, f)$$

$$= \alpha(t)\,\kappa(e, f) = \kappa(t, t_\alpha)\,\kappa(e, f)$$

$$= \kappa(t, \kappa(e, f) t_\alpha)$$

$$\Rightarrow \kappa(t, \underbrace{[e, f] - \kappa(e, f) t_\alpha}_{\in T}) = 0 \quad \forall\ t \in T$$

$\Rightarrow [e, f] - \kappa(e, f) t_\alpha = 0$ since $\kappa(\,,\,)|_T$ nondeg. //

(2) Suppose $\alpha(t_\alpha)=0$

$\alpha \neq 0 \Rightarrow t_\alpha \neq 0$

Choose $e \in L_\alpha$, $f \in L_{-\alpha}$ such that $\kappa(e,f)=1$.

$\xRightarrow{\text{by}(1)} \quad [e,f]=t_\alpha$

Now $[t_\alpha, e] = \alpha(t_\alpha)e = 0$

$[t_\alpha, f] = -\alpha(t_\alpha)f = 0$

$\Rightarrow H=\{e,f,t_\alpha\}$ is a Heisenberg subalg of L.

$\Rightarrow H$ nilp. $\Rightarrow H$ solvable

$\xRightarrow{\text{Lie's Thm}} \quad ad_H = \{ad_h \mid h \in H\} \subseteq gl(L)$

is simultaneously upper triangularizable.

$\Rightarrow ad_{t_\alpha} = ad_{[e,f]} = [ad\, e, ad\, f]$ is strictly upper triangular.

$\Rightarrow ad_{t_\alpha}$ is nilp.

But $t_\alpha \in T \Rightarrow ad_{t_\alpha}$ is semisimple

$\Rightarrow ad_{t_\alpha} = 0 \Rightarrow t_\alpha \in Z(L)=0$

$\Rightarrow t_\alpha = 0$ which is a contradiction. //

(3) & (4): Choose $0 \neq e_\alpha \in L_\alpha$.

Since $\alpha(t_\alpha) \neq 0$, we can and do choose $0 \neq f_\alpha \in L_{-\alpha}$ such that $\kappa(e_\alpha, f_\alpha) = \frac{2}{\alpha(t_\alpha)}$.

By (1), $[e_\alpha, f_\alpha] = \kappa(e_\alpha, f_\alpha) t_\alpha$

$$= \frac{2}{\alpha(t_\alpha)} t_\alpha := h_\alpha$$

$$\alpha(h_\alpha) = \alpha\left(\frac{2}{\alpha(t_\alpha)} t_\alpha\right) = \frac{2}{\alpha(t_\alpha)} \alpha(t_\alpha) = 2$$

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha) e_\alpha = 2 e_\alpha$$

$$[h_\alpha, f_\alpha] = -\alpha(h_\alpha) f_\alpha = -2 f_\alpha$$

$\Rightarrow \operatorname{span}\{e_\alpha, f_\alpha, h_\alpha\} \cong \mathfrak{sl}(2, \mathbb{C})$ //.