$\mathfrak{g}$ semisimple Lie alg. / $\mathbb{C}$

Fix any toral subalg. T of $\mathfrak{g}$
Δ = set of roots of T on $\mathfrak{g}$.

Then $\mathfrak{g} = L_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

By defn $T \subseteq L_0$ since T abelian.

Prop: $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$,

$\forall\ \alpha, \beta \in \Delta \cup \{0\}$.

Cor: (1) $\kappa(\ ,\ )\big|_{L_0 \times L_0}$ is nondeg.

(2) $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$.

Pf: Suppose $x \in \operatorname{rad} \kappa(\ ,\ )\big|_{L_0 \times L_0}$, $x \in L_0$

$\Rightarrow \kappa(x, y) = 0 \quad \forall\ y \in L_0$

$x \in L_0 \Rightarrow \kappa(x, \mathfrak{g}_\alpha) = 0 \quad \forall\ \alpha \in \Delta$
Let $z \in L = L_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$

$\Rightarrow z = y_1 + y_2$, $y_1 \in L_0$, $y_2 \in \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$

$\Rightarrow \kappa(x, z) = \kappa(x, y_1 + y_2) = \kappa(x, y_1) + \kappa(x, y_2) = 0$

$\Rightarrow x \in \operatorname{rad} \kappa(\ ,\ )$ which is nondeg.

$\Rightarrow x = 0$ proving (1).

(2) Let $\alpha \in \Delta \Rightarrow \mathfrak{g}_\alpha \neq 0$. Choose $0 \neq x \in \mathfrak{g}_\alpha$.

Suppose $\mathfrak{g}_{-\alpha} = 0$

Let $z \in L$. Then $z = y_1 + y_2$, $y_1 \in L_0$, $y_2 \in \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta$

$$\Rightarrow \kappa(x, y_1 + y_2) = \underset{\overset{\shortparallel}{0}}{\kappa(x, y_1)} + \underset{\overset{\shortparallel}{0}}{\kappa(x, y_2)}$$

by Prop since $\mathfrak{g}_{-\alpha} = 0$.

$\Rightarrow x \in \operatorname{rad} \kappa(\ ,\ ) \Rightarrow x = 0$ since $\kappa(,)$ nondeg.

which is a contradiction.

Hence $\mathfrak{g}_{-\alpha} \neq 0 \Rightarrow -\alpha \in \Delta$. //

<u>Defn</u>: A toral subalg T is max'l if T' toral subalg, $T' \supseteq T \Rightarrow T' = T$.

<u>Goal</u>:

<u>Thm</u>: T max'l toral subalg $\Rightarrow T = \mathfrak{g}_0$.

Lemma: For each $x \in L_0$ $\exists\, t_x \in T$, T max'l toral subalg,
and $x_\eta \in \mathfrak{g}_0$ s.t. $x = t_x + x_\eta$ where
$[t_x, x_\eta] = 0$, and $\mathrm{ad}\, x_\eta$ is nilp. on $\mathfrak{g}$, hence
on $\mathfrak{g}_0$.

Pf: $x \in \mathfrak{g}_0 \subseteq \mathfrak{g}$.

By AJCD, $\exists\, x_\sigma, x_\eta \in \mathfrak{g}$ s.t.
$\mathrm{ad}\, x_\sigma$ is semisimple, $\mathrm{ad}\, x_\eta$ is nilp.
and $[x_\sigma, x_\eta]$.

Recall $\mathrm{ad}\, x_\sigma = (\mathrm{ad}\, x)_s$, $\mathrm{ad}\, x_\eta = (\mathrm{ad}\, x)_n$
and $\mathrm{ad}\, x_\sigma$ & $\mathrm{ad}\, x_\eta$ are polys in $\mathrm{ad}\, x$
without constant term.

$x \in L_0 \Rightarrow \mathrm{ad}_x(t) = 0 \quad \forall\, t \in T$

$\Rightarrow (\mathrm{ad}\, x_\sigma)(t) = 0$ & $(\mathrm{ad}\, x_\eta)(t) = 0 \quad \forall\, t \in T$.

$\Rightarrow x_\sigma, x_\eta \in \mathfrak{g}_0$.

Consider $T' = \mathrm{span}\{T, x_\sigma\} \supseteq T$
and $\mathrm{ad}\, x_\sigma$ semisimple on $\mathfrak{g}$ &
$(\mathrm{ad}\, x_\sigma)(t) = 0 \quad \forall\, t \in T$

$\Rightarrow T'$ is a toral subalg. $\Rightarrow T' = T$ &
hence $t_x = x_\sigma \in T$. //

<u>Lemma:</u> T max'l toral subalg of $\mathfrak{g}$

$\Rightarrow \kappa(\,,\,)\big|_{T\times T}$ is nondeg.

<u>Pf:</u> Suppose ~~$x \in \kappa(\,,\,)$~~ $x \in \operatorname{rad} \kappa(\,,\,)\big|_{T\times T}$

$\Rightarrow \kappa(x,t) = 0 \;\forall\, t \in T$ (Note $x \in T$)

Let $y \in L_0 \Rightarrow y = t_y + y_\nu,\ t_y \in T$ &

$\operatorname{ad} y_\nu$ is nilp. & $[t_y, y_\nu] = 0,\ y_\nu \in L_0$

$$\kappa(x,y) = \kappa(x, t_y + y_\nu) = \underbrace{\kappa(x, t_y)}_{=0} + \kappa(x, y_\nu)$$

$$= \kappa(x, y_\nu)$$
$$= \operatorname{trace}(\operatorname{ad}_x \operatorname{ad} y_\nu)$$

Since $y_\nu \in L_0,\ x \in T_0$ we have $[x, y_\nu] = 0$

$\Rightarrow \operatorname{ad}_x$ & $\operatorname{ad}_{y_\nu}$ commute.

Since $x \in T \Rightarrow \operatorname{ad} x$ semisimple

$\operatorname{ad} y_\nu$ is nilp. & $\{\operatorname{ad} x, \operatorname{ad} y_\nu\}$ commuting

$\Rightarrow \operatorname{ad} x \operatorname{ad} y_\nu$ is nilp.

$\Rightarrow \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y_\nu) = \kappa(x, y_\nu) = 0$

$\Rightarrow \kappa(x,y) = 0 \Rightarrow x \in \operatorname{rad} \kappa(\,,\,)\big|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$

Since $\kappa(\ ,\ )\big|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ nondeg. we have $x = 0$

$\Rightarrow \kappa(\ ,\ )\big|_{T \times T}$ is nondeg. //

Lemma: T max'l toral subalg. of $\mathfrak{g}$.

$\Rightarrow T \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0$.

Pf: Let $x \in T \cap [\mathfrak{g}_0, \mathfrak{g}_0]$

$\Rightarrow x \in T,\ x \in [\mathfrak{g}_0, \mathfrak{g}_0] \Rightarrow x = \sum_i [x_i, y_i]$

$x_i, y_i \in \mathfrak{g}_0$

Let $t \in T$

$$\begin{aligned} \kappa(t, x) &= \kappa\Big(t, \sum_i [x_i, y_i]\Big) \\ &= \sum_i \kappa(t, [x_i, y_i]) \\ &= \sum_i \kappa(\underbrace{[t, x_i]}_{\overset{\shortparallel}{0}}, y_i) = 0 \end{aligned}$$

$\Rightarrow x \in \operatorname{rad} \kappa(\ ,\ )\big|_{T \times T}$ which is nondeg.

$\Rightarrow x = 0$ //.

Lemma: $\mathfrak{g}_0$ is abelian. (T max'l toral)

Pf: Let $x \in \mathfrak{g}_0 \Rightarrow x = t_x + x_n$

where $t_x \in T$, $\mathrm{ad}\, x_\nu$ nilp. & $[t_x, x_\nu]=0$.

$$\mathrm{ad}\, x\big|_{\mathfrak{g}_0} = \mathrm{ad}_{t_x + x_\nu}\big|_{\mathfrak{g}_0}$$

$$= \underbrace{\mathrm{ad}_{t_x}\big|_{\mathfrak{g}_0}}_{=0} + \mathrm{ad}\, x_\nu\big|_{\mathfrak{g}_0} = \mathrm{ad}\, x_\nu\big|_{\mathfrak{g}_0}$$

which is nilp.

$\Longrightarrow \mathrm{ad}\, x\big|_{\mathfrak{g}_0}$ nilp. $\forall\, x \in \mathfrak{g}_0$

$\Longrightarrow$ By Engel Thm. $\mathfrak{g}_0$ is nilp. ∎

Suppose $[\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$

$\mathfrak{g}_0$ nilp., $[\mathfrak{g}_0, \mathfrak{g}_0] \trianglelefteq \mathfrak{g}_0$

$\Longrightarrow Z(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$

Let $0 \neq y \in Z(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0]$

$y \in \mathfrak{g}_0 \Rightarrow y = t_y + y_\nu$, $t_y \in T$, $\mathrm{ad}\, y_\nu$ nilp.

$y_\nu \in \mathfrak{g}_0 \Longrightarrow [t_y, y_\nu] = 0$

$[T, \mathfrak{g}_0] = 0 \Rightarrow t_y \in Z(\mathfrak{g}_0)$

$\Rightarrow y_\nu = y - t_y \in Z(\mathfrak{g}_0)$.

$\kappa(y_\nu, x) = \operatorname{trace}(\operatorname{ad}_{y_\nu} \operatorname{ad}_x) = 0$

since $\operatorname{ad}_{y_\nu}$ nilp. and $[x, y_\nu] = 0$.

$\Longrightarrow$ $y_\nu = 0$ since $\kappa(\,,\,)|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ nondeg.

$\Longrightarrow$ $y = t_y \in T$, , $y \in Z(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0]$

$\Longrightarrow$ $y \in T \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0 \Rightarrow y = 0$ contradiction.

$\Longrightarrow$ $[\mathfrak{g}_0, \mathfrak{g}_0] = 0 \Rightarrow \mathfrak{g}_0$ abelian. //