$\mathfrak{g}$ semisimple Lie alg. $/\mathbb{C}$

A subalg. T of $\mathfrak{g}$ is a toral subalg. if ad_t semisimple (= diagonalizable) $\forall t \in T$.

Prop: T toral subalg. $\Rightarrow$ T abelian.

Pf: Let $t \in T$. ad_t semisimple on $\mathfrak{g}$

$\Rightarrow$ ad_t semisimple on T

$\Rightarrow$ $\exists$ ordered basis $\{s_1, s_2, \cdots, s_n\}$ of T such that $\underbrace{ad_t(s_i)}_{=[t,s_i]} = \alpha_i s_i$, $\alpha_i \in \mathbb{C}$

$$\Rightarrow [s_i, [t, s_i]] = 0 \Rightarrow \underbrace{[s_i, [s_i, t]]}_{= (ad_{s_i})^2 t} = 0$$

Since ad_{s_i} is semisimple, we have

$$\underbrace{ad_{s_i}(t)}_{=[s_i, t]} = 0, \ \forall i = 1, 2, \cdots, n.$$

$\Rightarrow$ $[t', t] = 0$ $\forall t' \in T$, $\forall t \in T$

$\Rightarrow$ T abelian. //

Fix toral subalg T for $\mathfrak{g}$.

T abelian $\Rightarrow \{ad_t \mid t \in T\}$ is a family of commuting semisimple operators on $\mathfrak{g}$.

$\Rightarrow \exists$ ordered basis $\mathcal{B} = \{x_1, x_2, \cdots, x_\ell\}$ for $\mathfrak{g}$ s.t.

$$[ad_t]_{\mathcal{B}} = \begin{pmatrix} \beta_1(t) & & & 0 \\ & \beta_2(t) & & \\ & & \ddots & \\ 0 & & & \beta_\ell(t) \end{pmatrix} \quad \forall\, t \in T$$

where $\beta_1, \beta_2, \cdots, \beta_\ell \in T^*$.

<u>Defn</u>: For $\alpha \in T^*$, define

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid ad_t(x) = \alpha(t)x \ \forall\, t \in T\}$$

A $0 \neq \alpha \in T^*$ is called a <u>root</u> of T on $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$. In such case $\mathfrak{g}_\alpha$ is called the α-root space.

Δ = set of roots of T on $\mathfrak{g}$.

<u>Ex(1)</u> $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$

$$T = \{\mathbb{C}\underbrace{(E_{11} - E_{22})}_{\text{"}t_1}\} = \langle t_1 \rangle = \text{span}\{t_1\}.$$

Then T is a toral subalg. of $\mathfrak{g}$ and $\dim T = 1$.

$\Rightarrow \dim T^* = 1$. Let $T^* = \text{span}\{\alpha\}$ where $\alpha(t_1) = 2$.

$$\Rightarrow \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t_1, x] = \alpha(t_1)x = 2x\}$$

$$\mathfrak{g}_{-\alpha} = \{x \in \mathfrak{g} \mid [t_1, x] = -2x\}$$

$$\mathfrak{g} = \text{span}\{t_1 = E_{11} - E_{22},\ t_2 = E_{22} - E_{33},\ E_{12}, E_{23},$$
$$E_{32}, E_{21}, E_{13}, E_{31}\}$$

$$[t_1, t_1] = 0, \quad [t_1, t_2] = 0$$
$$[t_1, E_{12}] = 2E_{12}, \quad [t_1, E_{23}] = -E_{23},$$
$$[t_1, E_{32}] = E_{32}, \quad [t_1, E_{21}] = -2E_{21}$$
$$[t_1, E_{13}] = E_{13}, \quad [t_1, E_{31}] = -E_{31}$$
$$\alpha(t_1) = 2 \Rightarrow \tfrac{1}{2}\alpha(t_1) = 1$$

$$\Rightarrow \quad \mathfrak{g}_\alpha = \text{span}\{E_{12}\}, \quad \mathfrak{g}_{-\alpha} = \text{span}\{E_{21}\}$$

$$\mathfrak{g}_{\frac{1}{2}\alpha} = \text{span}\{E_{32}, E_{13}\}$$

$$\mathfrak{g}_{-\frac{1}{2}\alpha} = \text{span}\{E_{23}, E_{31}\}, \quad \mathfrak{g}_0 = \text{span}\{t_1, t_2\}$$

$$\Delta = \{\pm\alpha, \pm \tfrac{1}{2}\alpha\}$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\beta\in\Delta} \mathfrak{g}_\beta \, , \quad T \subsetneq \mathfrak{g}_0$$

<u>Prop:</u> T toral subalg. of $\mathfrak{g}$, Δ = set of roots of T on $\mathfrak{g}$. Then

(1) $T \subseteq \mathfrak{g}_0$ (since T abelian)

(2) $\mathfrak{g}_0 = C_{\mathfrak{g}}(T)$ (by defn of $\mathfrak{g}_0$)

(3) $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ (by defn)

(4) $\forall\, \alpha, \beta \in \Delta \cup \{0\}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

<u>Pf (4)</u> Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $t \in T$

$$ad_t([x,y]) \overset{?}{=} (\alpha+\beta)(t)[x,y]$$

$$x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta \Rightarrow \begin{array}{l} ad_t(x) = \alpha(t)x \\ ad_t(y) = \beta(t)y \end{array}$$

$$\begin{aligned} \Rightarrow ad_t[x,y] &= [ad_t(x), y] + [x, ad_t(y)] \\ &= \alpha(t)[x,y] + \beta(t)[x,y] \\ &= (\alpha+\beta)(t)[x,y]. \qquad // \end{aligned}$$

<u>Prop</u>: $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right)$

<u>Pf</u>: Enough to show that there exists no dependency relations.

Suppose there are dependency relations. Then choose a minimal dependency relation:

$(*) \quad x_\alpha + x_\beta + \cdots\cdots = 0 \quad , \alpha, \beta \in \Delta \cup \{0\}.$

with atleast two nonzero terms

$$x_\alpha \in \mathfrak{g}_\alpha , \ x_\beta \in \mathfrak{g}_\beta ; \quad \alpha \neq \beta .$$

Apply ad_t, $t \in T$ to $(*)$

$(**) \quad \alpha(t) x_\alpha + \beta(t) x_\beta + \cdots\cdots = 0$

Multiplying $(*)$ by $\alpha(t)$ we have

$(***) \quad \alpha(t) x_\alpha + \alpha(t) x_\beta + \cdots\cdots = 0$

Substracting $(***)$ from $(**)$ get

$$(\beta(t) - \alpha(t)) x_\beta + \cdots\cdots = 0$$

Since $\alpha \neq \beta$, we can choose $t \in T$ such that $\beta(t) \neq \alpha(t)$. Then we have a smaller dependency relation than $(*)$ which is a contradiction. //

Recall the Killing form $\kappa(\,,\,): \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is nondegenerate.

Prop: $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$, $(\alpha, \beta \in \Delta \cup \{0\})$.

Pf: Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $t \in T$

Then $\alpha(t)\kappa(x, y) = \kappa(\alpha(t)x, y)$

$$= \kappa([t, x], y) = -\kappa([x, t], y)$$
$$= -\kappa(x, [t, y]) = -\beta(t)\kappa(x, y)$$

$\Rightarrow (\alpha(t) + \beta(t))\kappa(x, y) = 0 \quad \forall\, t \in T$

Suppose $\beta \neq -\alpha$. Then $\exists\, t \in T$ such that $\alpha(t) + \beta(t) \neq 0 \Rightarrow \kappa(x, y) = 0 \;\forall\, x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$. //

Cor: $\kappa(x, y) \neq 0$ for some $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$.

Pf: This follows since $\kappa(\,,\,)$ is nondegenerate, and the above Prop. //