<u>9/19</u>

$\lambda \in P^+ \qquad V(\lambda) \quad \mathfrak{sl}(n, F)$-module

- Find all dominant wts of $V(\lambda)$
- μ is dominant wt. of $V(\lambda)$. Then

  $\dim V(\lambda)_\mu$ = # semistandard tableaux of shape λ and wt. μ.

  $= K_{\lambda,\mu}$ (Kostka number)

- If ν is any wt. of $V(\lambda)$, Then

  $\exists\ \sigma \in S_n$ s.t. $\sigma\nu$ is a dominant wt.

- $\dim V(\lambda)_\nu = \dim V(\lambda)_{\sigma\nu}$

<u>Ex(1)</u> $\lambda = \omega_1 + \omega_2 = 2\varepsilon_1 + \varepsilon_2 \qquad \mathfrak{g} = \mathfrak{sl}(3, F)$

- Dominant wts of $V(\lambda)$:

  $\lambda = 2\varepsilon_1 + \varepsilon_2 \quad , \quad \mu = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \Rightarrow K_{\lambda,\lambda} = 1$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \Rightarrow K_{\lambda,\mu} = 2.$$

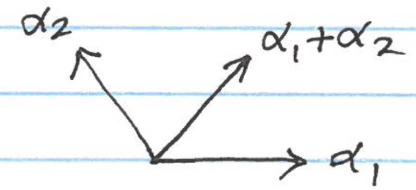
Weights of $V(\lambda)$ conjugate to λ:

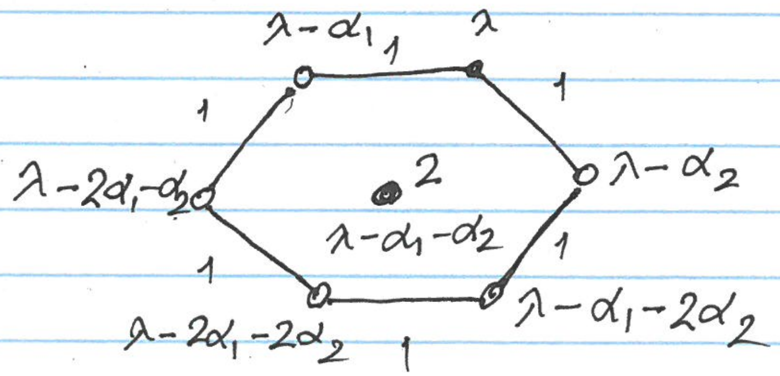
$$S_3 = \{(1), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$$

$$K_{\lambda,\nu} = 1 \begin{cases} (1) & \lambda = 2\varepsilon_1 + \varepsilon_2 \\ (2) & \varepsilon_1 + 2\varepsilon_2 = \lambda - \varepsilon_1 + \varepsilon_2 = \lambda - \alpha_1 \\ (3) & \varepsilon_2 + 2\varepsilon_3 = \lambda - 2\varepsilon_1 + 2\varepsilon_3 = \lambda - 2\alpha_1 - 2\alpha_2 \\ (4) & 2\varepsilon_1 + \varepsilon_3 = \lambda - \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_2 \\ (5) & 2\varepsilon_2 + \varepsilon_3 = \lambda - 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - 2\alpha_1 - \alpha_2 \\ (6) & \varepsilon_1 + 2\varepsilon_3 = \lambda - \varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 = \lambda - \alpha_1 - 2\alpha_2 \end{cases}$$

Weights of $V(\lambda)$ conjuge to μ:

$K_{\lambda,\mu} = 2$ (7) $\mu = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - \varepsilon_1 + \varepsilon_3 = \lambda - \alpha_1 - \alpha_2$





<u>Ex(2)</u> $\lambda = \omega_1 + 2\omega_2 = 3\varepsilon_1 + 2\varepsilon_2$, $\mathfrak{g} = \mathfrak{sl}(3, F)$

Dominant weights of $V(\lambda)$:

- $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ $\quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \Rightarrow K_{\lambda,\lambda} = 1$

- $\cancel{2\varepsilon_1 + 3\varepsilon_2 = \lambda}$

- $\mu_1 = 3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_2$

  $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \Rightarrow K_{\lambda,\mu_1} = 1$

- $\mu_2 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 = \lambda - \alpha_1 - \alpha_2$

  $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \Rightarrow K_{\lambda,\mu_2} = 2.$

Weights conjugate to λ:

- $\lambda = 3\varepsilon_1 + 2\varepsilon_2$
- $2\varepsilon_1 + 3\varepsilon_2 = \lambda - \varepsilon_1 + \varepsilon_2 = \lambda - \alpha_1$
- $3\varepsilon_1 + 2\varepsilon_3 = \lambda - 2\varepsilon_2 + 2\varepsilon_3 = \lambda - 2\alpha_2$
- $2\varepsilon_2 + 3\varepsilon_3 = \lambda - 3\varepsilon_1 + 3\varepsilon_3 = \lambda - 3\alpha_1 - 3\alpha_2$
- $3\varepsilon_2 + 2\varepsilon_3 = \lambda - 3\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 = \lambda - 3\alpha_1 - 2\alpha_2$
- $2\varepsilon_1 + 3\varepsilon_3 = \lambda - \varepsilon_1 - 2\varepsilon_2 + 3\varepsilon_3 = \lambda - \alpha_1 - 3\alpha_2$

Weights conjugate to μ_1:

- $\mu_1 = 3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_2$
- $\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 = \mu_1 - 2\varepsilon_1 + 2\varepsilon_2 = \lambda - 2\alpha_1 - \alpha_2$
- $\varepsilon_1 + \varepsilon_2 + 3\varepsilon_3 = \mu_1 - 2\varepsilon_1 + 2\varepsilon_3 = \lambda - 2\alpha_1 - 3\alpha_2$

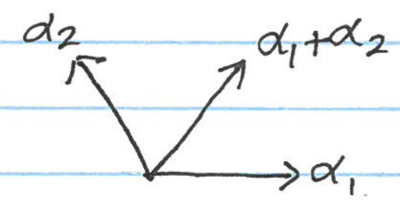
Weights conjugate to μ_2:

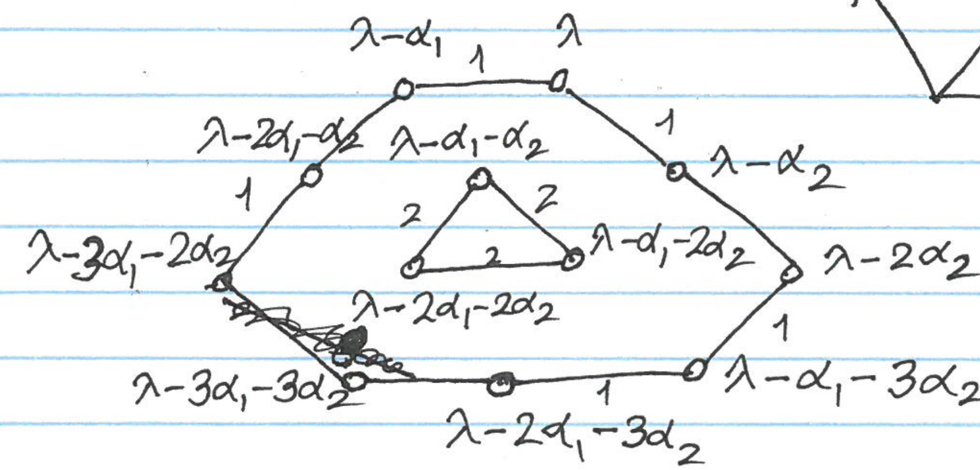
- $\mu_2 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 = \lambda - \alpha_1 - \alpha_2$
- $2\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 = \mu_2 - \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_1 - 2\alpha_2$
- $\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 = \mu_2 - \varepsilon_1 + \varepsilon_3 = \lambda - 2\alpha_1 - 2\alpha_2$

$\dim V(\lambda) = 9 + 2(3) = 15$

Remark: $\lambda = m_1\omega_1 + m_2\omega_2$, $\mathfrak{g} = \mathfrak{sl}(3, F)$

$$\dim V(\lambda) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$$





Humphreys, P115, $V(4\omega_1 + 3\omega_2)$, $\mathfrak{g} = \mathfrak{sl}(3, F)$

Structure of fin. dim'l semisimple Lie algebra $\mathfrak{g}$:

F alg. closed & char $F=0$ $(F=\mathbb{C})$.

$\mathfrak{g}$ semisimple $\Leftrightarrow$ $\mathrm{rad}(\mathfrak{g})=0$.

$\mathcal{K}(\ ,\ )$ Killing form is nondegenerate.

<u>Defn:</u> A subalgebra T of $\mathfrak{g}$ is a <u>toral subalgebra</u> if ad_t is semisimple $\forall\, t\in T$.

<u>Prop:</u> $\mathfrak{g}$ contains a nonzero toral subalgebra.

<u>Pf:</u> Let $0\neq x\in\mathfrak{g}$.

$x = x_s + x_n$ AJCD

where • ad_{x_s} semisimple, ad_{x_n} nilp.

• $[x_s, x_n]=0$

Suppose $x_s=0$ $\forall\, x\in\mathfrak{g}$

$\Rightarrow x=x_n \Rightarrow \mathrm{ad}_x=\mathrm{ad}_{x_n}$ nilp. $\forall\, x\in\mathfrak{g}$

$\overset{\text{Engel}}{\Longrightarrow}$ $\mathfrak{g}$ nilpotent which is a contradiction.

$\Rightarrow \exists\, x\in\mathfrak{g}$ s.t. $x_s\neq 0 \Rightarrow T=\mathbb{C}x_s$ toral subalg. //

Ex (3) $\lambda = 3\omega_1 + 2\omega_2 = \{5, 2\}$, $g = sl(3, F)$

Dominant weights of $V(\lambda)$:

(1) $\lambda = 5\varepsilon_1 + 2\varepsilon_2$: $\begin{array}{ccccc} 1&1&1&1&1 \\ 2&2 \end{array}$, $K_{\lambda,\lambda} = 1$

(2) $\mu_1 = 4\varepsilon_1 + 3\varepsilon_2$: $\begin{array}{ccccc} 1&1&1&1&2 \\ 2&2 \end{array}$, $K_{\lambda,\mu_1} = 1$

$= \lambda - \alpha_1$

(3) $\mu_2 = 4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$: $\begin{array}{ccccc} 1&1&1&1&3 \\ 2&2 \end{array}$ $\begin{array}{ccccc} 1&1&1&1&2 \\ 2&3 \end{array}$

$= \lambda - \alpha_1 - \alpha_2$

$\Rightarrow K_{\lambda,\mu_2} = 2$

(4) $\mu_3 = 3\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3$: $\begin{array}{ccccc} 1&1&1&2&2 \\ 2&3 \end{array}$ $\begin{array}{ccccc} 1&1&1&2&3 \\ 2&2 \end{array}$

$= \lambda - 2\alpha_1 - \alpha_2$

$\Rightarrow K_{\lambda,\mu_3} = 2$

(5) $\mu_4 = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$: $\begin{array}{ccccc} 1&1&1&2&2 \\ 3&3 \end{array}$ $\begin{array}{ccccc} 1&1&1&3&3 \\ 2&2 \end{array}$

$= \lambda - 2\alpha_1 - 2\alpha_2$

$\begin{array}{ccccc} 1&1&1&2&3 \\ 2&3 \end{array}$

$\Rightarrow K_{\lambda,\mu_4} = 3$

(6) $\mu_{\bullet} = 5\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ $\begin{array}{ccccc} 1&1&1&1&1 \\ 2&3 \end{array}$

$= \lambda - \alpha_2$

, $K_{\lambda,\mu_{\bullet}} = 1$

Weights conjugate to λ:

$5\varepsilon_1 + 2\varepsilon_2 = \lambda$

$2\varepsilon_1 + 5\varepsilon_2 = \lambda - 3\alpha_1$

$5\varepsilon_1 + 2\varepsilon_3 = \lambda - 2\alpha_2$

$2\varepsilon_2 + 5\varepsilon_3 = \lambda - 5\alpha_1 - 5\alpha_2$

$5\varepsilon_2 + 2\varepsilon_3 = \lambda - 5\alpha_1 - 2\alpha_2$

$2\varepsilon_1 + 5\varepsilon_3 = \lambda - 3\alpha_1 - 5\alpha_2$

Weights conjugate to μ_1:

$$4\varepsilon_1+3\varepsilon_2 = \lambda-\alpha_1$$
$$3\varepsilon_1+4\varepsilon_2 = \lambda-2\alpha_1$$
$$4\varepsilon_1+3\varepsilon_3 = \lambda-\alpha_1-3\alpha_2$$
$$3\varepsilon_2+4\varepsilon_3 = \lambda-5\alpha_1-4\alpha_2$$
$$4\varepsilon_2+3\varepsilon_3 = \lambda-5\alpha_1-3\alpha_2$$
$$3\varepsilon_1+4\varepsilon_3 = \lambda-2\alpha_1-4\alpha_2$$

Weights conjugate to μ_2:

$$4\varepsilon_1+2\varepsilon_2+\varepsilon_3 = \lambda-\alpha_1-\alpha_2$$
$$2\varepsilon_1+4\varepsilon_2+\varepsilon_3 = \lambda-3\alpha_1-\alpha_2$$
$$4\varepsilon_1+\varepsilon_2+2\varepsilon_3 = \lambda-\alpha_1-2\alpha_2$$
$$\varepsilon_1+2\varepsilon_2+4\varepsilon_3 = \lambda-4\alpha_1-4\alpha_2$$
$$\varepsilon_1+4\varepsilon_2+2\varepsilon_3 = \lambda-4\alpha_1-2\alpha_2$$
$$2\varepsilon_1+\varepsilon_2+4\varepsilon_3 = \lambda-3\alpha_1-4\alpha_2$$

Weights conjugate to μ_3:

$$3\varepsilon_1+3\varepsilon_2+\varepsilon_3 = \lambda-2\alpha_1-\alpha_2$$
$$3\varepsilon_1+\varepsilon_2+3\varepsilon_3 = \lambda-2\alpha_1-3\alpha_2$$
$$\varepsilon_1+3\varepsilon_2+3\varepsilon_3 = \lambda-4\alpha_1-3\alpha_2$$

Weights conjugate to μ_4:

$$3\varepsilon_1+2\varepsilon_2+2\varepsilon_3 = \lambda-2\alpha_1-2\alpha_2$$
$$2\varepsilon_1+3\varepsilon_2+2\varepsilon_3 = \lambda-3\alpha_1-2\alpha_2$$
$$2\varepsilon_1+2\varepsilon_2+3\varepsilon_3 = \lambda-3\alpha_1-3\alpha_2$$

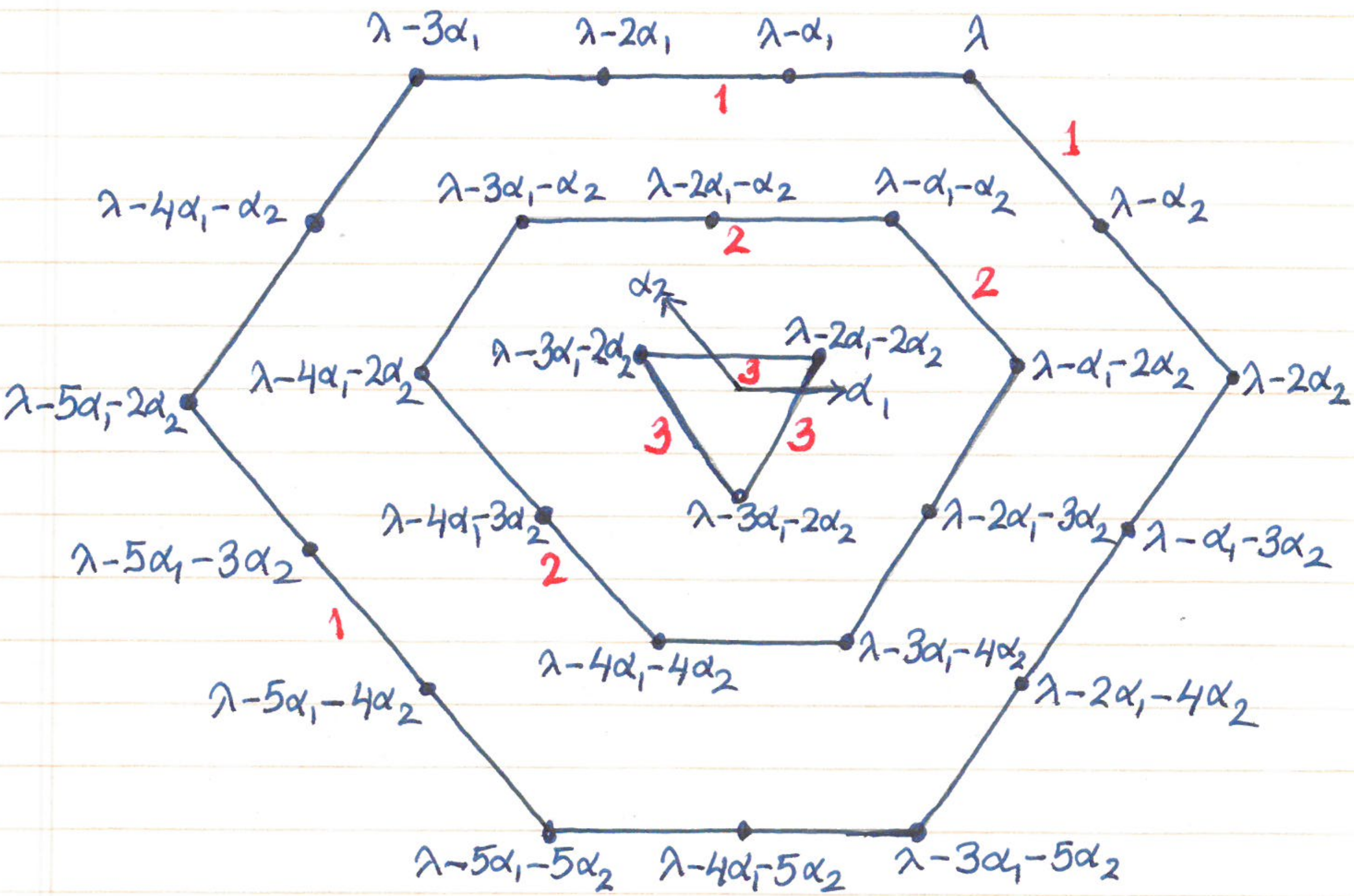
Weights conjugate to μ :

$$5\varepsilon_1+\varepsilon_2+\varepsilon_3=\lambda-\alpha_2$$
$$\varepsilon_1+5\varepsilon_2+\varepsilon_3=\lambda-4\alpha_1-\alpha_2$$
$$\varepsilon_1+\varepsilon_2+5\varepsilon_3=\lambda-4\alpha_1-5\alpha_2$$

Remark: $V(m_1\omega_1+m_2\omega_2)$ $sl(3,F)$-module

$$\dim V(m_1\omega_1+m_2\omega_2)=\tfrac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$$

$$\therefore \dim V(3\omega_1+2\omega_2)=\tfrac{1}{2}(4)(3)(7)=42$$
$$=6+6+3+12+6+9.$$



(See Humphreys, Page 115 for $V(4\omega_1+3\omega_2)$.)