Ex(1) $\lambda = 3\omega_1 + 2\omega_2$, $V(\lambda)$ $\mathfrak{sl}(3, F)$-module

$= 5\varepsilon_1 + 2\varepsilon_2$

$= \{5, 2\}$

The dominant weights of $V(\lambda)$ are:

$\lambda = 5\varepsilon_1 + 2\varepsilon_2$, $\mu = 5\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_2$

$\mu_1 = 4\varepsilon_1 + 3\varepsilon_2 = \lambda - \varepsilon_1 + \varepsilon_2 = \lambda - \alpha_1$

$\mu_2 = 4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 = \lambda - \varepsilon_1 + \varepsilon_3 = \lambda - \alpha_1 - \alpha_2$

$\mu_3 = 3\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 = \lambda - 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - 2\alpha_1 - \alpha_2$

$\mu_4 = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 = \lambda - 2\varepsilon_1 + 2\varepsilon_3 = \lambda - 2\alpha_1 - 2\alpha_2$

The weight of a semistandard tableau μ is given by

$$w(\mu) = \sum_{j=1}^{n} (\# j\text{'s in } F(\mu))\,\varepsilon_j$$

Ex(2)

(i) $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ & $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ have wt. $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$

(ii) $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$ & $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$

have wt. $4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$.

$\lambda \in P^+$, $V(\lambda)$ $\mathfrak{g} = \mathfrak{sl}(n, F)$-module

$\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0\} \vdash m$.

<u>Thm:</u> $V(\lambda)$ has a basis parametrized by semistandard tableaux of shape λ with fillings by numbers $1, 2, \ldots, n$.

<u>Ex (1)</u> $V(2\omega_1)$ $\mathfrak{sl}(3, F)$-module

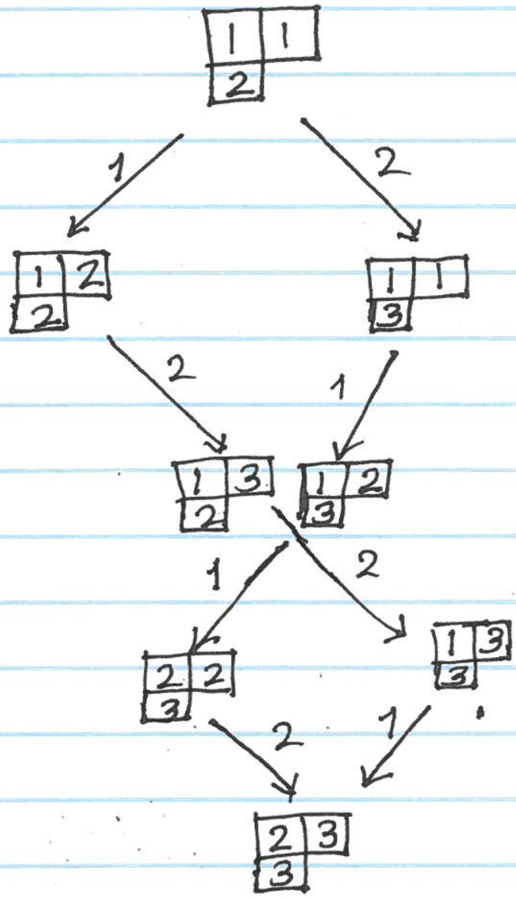
$2\omega_1 = 2\varepsilon_1$ $\qquad$ $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$

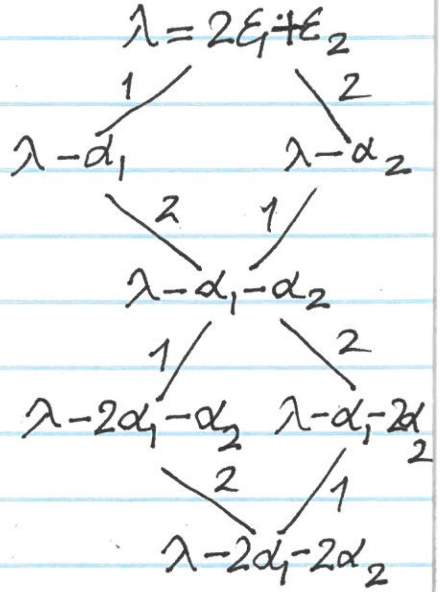
$$\begin{tikzcd}[column sep=small]
& \boxed{1\,|\,1} \arrow[d, "1"'] & \\
& \boxed{1\,|\,2} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\boxed{2\,|\,2} \arrow[dr, "2"'] & & \boxed{1\,|\,3} \arrow[dl, "1"] \\
& \boxed{2\,|\,3} \arrow[d, "2"] & \\
& \boxed{3\,|\,3} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
& \lambda = 2\varepsilon_1 \arrow[d, "1"'] & \\
& \lambda - \alpha_1 \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\lambda - 2\alpha_1 \arrow[dr, "2"'] & & \lambda - \alpha_1 - \alpha_2 \arrow[dl, "1"] \\
& \lambda - 2\alpha_1 - \alpha_2 \arrow[d, "2"] & \\
& \lambda - 2\alpha_1 - 2\alpha_2 &
\end{tikzcd}$$

$\dim V(2\omega_1) = 6$

Also $\dim V(2\omega_1)_\mu = 1$ for all wts μ of $V(\lambda)$

Ex(2) $V(\omega_1+\omega_2) = V(2\varepsilon_1+\varepsilon_2)$ $sl(3,F)$-module

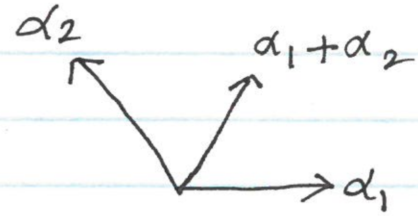


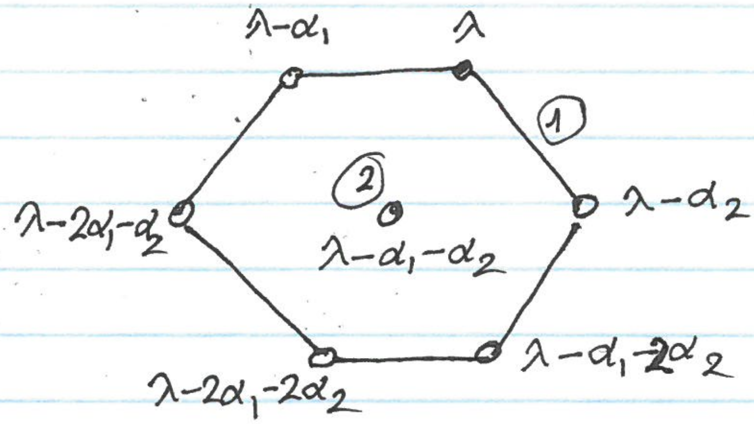


$\dim V(\omega_1+\omega_2) = 8$

$\dim V(\omega_1+\omega_2)_{\lambda-\alpha_1-\alpha_2} = 2$

$\dim V(\omega_1+\omega_2)_{\mu} = 1$, $\mu \neq \lambda-\alpha_1-\alpha_2$ & μ a wt.





$\lambda \in P^+$, $V(\lambda)$ $\mathfrak{sl}(n,F)$-module

μ is a wt. of $V(\lambda)$ ($\Rightarrow \dim V(\lambda)_\mu \neq 0$)

<u>Thm:</u> $\dim V(\lambda)_\mu =$ # semistandard tableaux of shape λ and wt. μ.

<u>Ex (1)</u> $\lambda = 3\omega_1 + 2\omega_2 = 5\varepsilon_1 + \varepsilon_2$, $n=3$

$\mu = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$

$\dim V(\lambda)_\mu = ?$

1	1	1	2	2
3	3			

1	1	1	2	3
2	3			

1	1	1	3	3
2	2			

$\Rightarrow \dim V(\lambda)_\mu = 3.$

$\lambda \in P^+$, $V(\lambda)$ $sl(n, F)$-module

$$\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_{l(\lambda)} \varepsilon_{l(\lambda)}, \quad l(\lambda) \leq n.$$

μ be any weight of $V(\lambda)$

$$\Rightarrow \mu = \lambda - \sum_{i=1}^{n-1} m_i \alpha_i, \quad m_i \in \mathbb{Z}_{\geq 0}$$

Recall $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$$\begin{aligned} \Rightarrow -\alpha_i &= \varepsilon_{i+1} - \varepsilon_i \\ &= \varepsilon_{i+1} + (\varepsilon_1 + \cdots + \varepsilon_{i-1} + \varepsilon_{i+1} + \cdots + \varepsilon_n) \\ &= \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{i-1} + 2\varepsilon_{i+1} + \varepsilon_{i+2} + \cdots + \varepsilon_n \end{aligned}$$

$$\Rightarrow \mu_1 = \sum_{j=1}^{n} \theta_j \varepsilon_j, \quad \theta_j \in \mathbb{Z}_{\geq 0}$$

Recall Weyl group of $sl(n, F)$:

$$W = \langle r_1, r_2, \ldots, r_{n-1} \rangle = S_n$$

where $r_i = (i, i+1)$.

$$\forall \sigma \in S_n, \quad \sigma\mu = \sum_{j=1}^{n} \theta_j \varepsilon_{\sigma(j)}$$

$\Rightarrow \exists\ \alpha \in S_n$ such that $\alpha(\mu) \in P^+$

<u>Thm:</u> $\lambda \in P^+$, $V(\lambda)$ irred. $sl(n, F)$
μ any weight of $V(\lambda)$ and $\alpha \in S_n$.
Then

$$\dim V(\lambda)_{\alpha(\mu)} = \dim V(\lambda)_\mu.$$