Semistandard Tableau: ($\mathfrak{g} = \mathfrak{sl}(n, F)$)

A semistandard tableau of shap $\lambda \vdash m$ is a filling of $F(\lambda)$ with numbers $\{1, 2, \dots, n\}$ in such a way that the entries weakly increase across rows from left to right & strictly increase across column top to bottom.

$\mathfrak{g} = \mathfrak{sl}(n, F)$, $\lambda \in P^+$, $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0\}$

$\lambda \vdash m$. $V(\lambda)$ irred. fin-dim'l $\mathfrak{g}$-module.

__Thm__: $V(\lambda)$ has a basis indexed by semistandard tableaux of shape λ.

__Ex (1)__ $\mathfrak{g} = \mathfrak{sl}(3, F)$, $\lambda = \omega_1 + \omega_2 = \{2, 1\}$

The basis of $V(\lambda)$ is indexed by semistandard tableaux:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$\Rightarrow \dim(V(\lambda)) = 8.$

$\lambda \in P^+$, $\lambda \vdash m = |\lambda|$, $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0\}$

$V(\lambda)$ irred. fin. dim'l $\mathfrak{g}$-module.

Suppose $\mu = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\ell(\mu)} > 0\}$

Define $\rho_i(\mu) = \mu_1 + \mu_2 + \cdots + \mu_i$

called the ith partial sum of μ.

Note $\rho_{\ell(\mu)} = |\mu| = \rho_i(\mu) \quad i \geq \ell(\mu)$

Dominance order (or Snapper order):

For $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0\}$ and $\mu = \{\mu_1 \geq \cdots \geq \mu_{\ell(\mu)} > 0\}$ we say $\lambda \geq \mu$ (i.e. λ dominates μ) if $\rho_i(\lambda) \geq \rho_i(\mu)$ $\forall\, i \geq 1$.

<u>Thm:</u> For $\lambda \in P^+$, $V(\lambda)$ $\mathfrak{g}$-module.

The set of dominant weights of $V(\lambda)$ is

$$\{\mu = \{\mu_1 \geq \cdots \geq \mu_{\ell(\mu)} > 0\} \mid \mu \vdash |\lambda|, \lambda \geq \mu, \ell(\mu) \leq n\}$$

Ex(2): $\lambda = \omega_1 + \omega_2 = \{2, 1\}$, $\mathfrak{g} = \mathfrak{sl}(3, F)$

The dominant weights of $V(\lambda)$ are:

$$\lambda = \{2, 1\}, \quad \mu = \{1, 1, 1\}$$

Ex(3) $\lambda = 3\omega_1 + 2\omega_2 = \{5, 2\}$, $\mathfrak{g} = \mathfrak{sl}(3, F)$

The dominant weights of $V(\lambda)$ are:

$$\lambda = \{5, 2\}, \quad \mu = \{4, 3\}, \quad \mu_1 = \{4, 2, 1\}$$

$$\mu_2 = \{3, 3, 1\}, \quad \mu_3 = \{3, 2, 2\}, \quad \mu_4 = \{5, 1, 1\}$$

Recall that if μ is a weight of $V(\lambda)$ then $\mu = \lambda - \sum_{i=1}^{n-1} m_i \alpha_i$, $m_i \in \mathbb{Z}_{\geq 0}$

$\mathfrak{g} = \mathfrak{sl}(n, F)$, $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_{\ell(\lambda)} \varepsilon_{\ell(\lambda)}$

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$$

$$\Rightarrow -\alpha_i = -(\varepsilon_i - \varepsilon_{i+1}) = \varepsilon_{i+1} - \varepsilon_i$$

$$= \varepsilon_{i+1} + (\varepsilon_1 + \cdots + \varepsilon_{i-1} + \varepsilon_{i+1} + \cdots + \varepsilon_n)$$

$$= \varepsilon_1 + \cdots + \varepsilon_{i-1} + 2\varepsilon_{i+1} + \cdots + \varepsilon_n$$

$\Rightarrow \mu = k_1 \varepsilon_1 + k_2 \varepsilon_2 + \cdots + k_n \varepsilon_n, \quad k_i \in \mathbb{Z}_{\geq 0}$.

$W = S_n$, $\sigma \in S_n$ ~~such that~~

$$\sigma \mu = k_1 \varepsilon_{\sigma(1)} + k_2 \varepsilon_{\sigma(2)} + \cdots + k\, \varepsilon_{\sigma(n)}$$

$\Rightarrow \exists\ \sigma \in S_n$ such that

$\lambda = \sigma\mu$ is a dominant weight.

i.e. Any weight of $V(\lambda)$ is Weyl group conjugate to a dominant weight.

<u>Ex (4)</u> $\lambda = \omega_1 + \omega_2 = \{2, 1\}$, $g = sl(3, F)$

Dominant weights of $V(\lambda)$:

$$\lambda = 2\varepsilon_1 + \varepsilon_2 = (\alpha_1 + \alpha_2), \qquad \mu = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - \varepsilon_1 + \varepsilon_3 = \lambda - (\varepsilon_1 - \varepsilon_3) = \lambda - \alpha_1 - \alpha_2$$

$W = S_3 = \{(1), (12), (2,3), (13), (123), (132)\}$

Weights conjugate to λ:

$\lambda = 2\varepsilon_1 + \varepsilon_2 \quad (= \alpha_1 + \alpha_2)$

- $\varepsilon_1 + 2\varepsilon_2 = \lambda - \varepsilon_1 + \varepsilon_2 = \lambda - \alpha_1$

- $2\varepsilon_1 + \varepsilon_3 = \lambda - \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_2$
- $\varepsilon_2 + 2\varepsilon_3 = \lambda - 2\varepsilon_1 + 2\varepsilon_3 = \lambda - 2\alpha_1 - 2\alpha_2$
- $2\varepsilon_2 + \varepsilon_3 = \lambda - 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - (\varepsilon_1 - \varepsilon_2) - (\varepsilon_1 - \varepsilon_3)$
  $= \lambda - 2\alpha_1 - \alpha_2$
- $\varepsilon_1 + 2\varepsilon_3 = \lambda - \varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 = \lambda - \alpha_1 - 2\alpha_2$

Weights conjugate to μ;

$$\mu = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda - \alpha_1 - \alpha_2$$

$$\begin{tikzcd}[column sep=small]
 & \lambda = \alpha_1 + \alpha_2 \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\alpha_2 = \lambda - \alpha_1 \arrow[dr, "2"'] & & \lambda - \alpha_2 = \alpha_1 \arrow[dl, "1"] \\
 & \lambda - \alpha_1 - \alpha_2 = 0 \arrow[dl, "1"'] \arrow[dr, "2"] & \\
-\alpha_1 = \lambda - 2\alpha_1 - \alpha_2 \arrow[dr, "2"'] & & \lambda - \alpha_1 - 2\alpha_2 = -\alpha_2 \arrow[dl, "1"] \\
 & \lambda - 2\alpha_1 - 2\alpha_2 = -\alpha_1 - \alpha_2 &
\end{tikzcd}$$

$$\dim V(\lambda)_0 = \dim \mathfrak{h} = 2$$

$$\dim V(\lambda)_\mu = 1, \quad \mu \neq 0.$$

Defn: The *weight* of a semistandard tableau μ by

$$w(\mu) = \sum_{j=1}^{n} (\#j\text{'s in } F(\mu))\, \varepsilon_j.$$

Ex(5) The weight of

$$\mu = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

is $w(\mu) = 4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$.