$\mathfrak{g} = sl(n, F)$, char. $F = 0$

$P^+ = \{ \lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq n-1 \}$

$V(\lambda)$ irred. fin. dim'l $\mathfrak{g}$-module with highest wt. vector v_λ.

$\mu \in P^+$, $V(\mu)$.

By Weyl's Thm, $V(\lambda) \otimes V(\mu)$ is completely reducible.

Lattice permutation:

A word $w = b_1 b_2 \cdots b_q$ of positive integers is a lattice permutation if for each $k \in \mathbb{Z}_{\geq 0}$ the

(# occurances of k) $\geq$ (# occurances of $k+1$)

in each subword $b_1 b_2 \cdots b_p$, $1 \leq p \leq q$.

Ex: 1212, 1122 are lattice permutations whereas 2211 is not.

Let $w = b_1 b_2 \cdots b_k$ be a lattice permutation and

$\nu_k = \#$ of occurances of k in w.

Then $\nu_1 \geq \nu_2 \geq \nu_3 \geq \cdots$

$\nu = \{\nu_1 \geq \nu_2 \geq \cdots\}$ is called the weight of w.

Thm: Littlewood - Richardson rule;

$$V(\lambda) \otimes V(\mu) = \bigoplus \sum_{\substack{\nu \vdash |\lambda| + |\mu| \\ l(\nu) \leq n}} c^{\nu}_{\lambda,\mu} V(\nu)$$

where $c^{\nu}_{\lambda,\mu}$ = # of lattice permutations of shap ν/λ and weight μ, as we read 'top to bottom' & 'right to left'.
= # of lattice permutations (as we read 'top to bottom' & 'right to left') of weight μ which give semistandard tableaux (or column strict tableaux) when inserted into ν/λ.

Here ν/λ is the diagram obtained by deleting the boxes in $F(\lambda)$ from $F(\nu)$.

$c^{\nu}_{\lambda,\mu}$ are called Littlewood-Richardson coefficients.

Ex(1) $\mathfrak{g} = \mathfrak{sl}(3, F)$, $\lambda = \omega_1 + \omega_2 = 2\varepsilon_1 + \varepsilon_2 = \{2,1\}$ and $\mu = 2\omega_2 = 2\varepsilon_1 + 2\varepsilon_2 = \{2^2\}$.

$$F(\lambda) \otimes F(\mu) = \begin{ytableau} \ & \ \\ \ \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau}$$

$$= \begin{ytableau} \ & \ & 1 & 1 \\ \ & 2 \\ 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & 1 & 1 \\ \ & 2 & 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & 1 \\ \ & 1 & 2 \\ 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & 1 \\ \ & 1 \\ 2 & 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & 1 \\ \ & 2 \\ 1 \\ 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ \\ \ & 1 \\ 1 & 2 \\ 2 \end{ytableau}$$

$$\Rightarrow V(\lambda) \otimes V(\mu) \cong V(4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3) \oplus V(4\varepsilon_1 + 3\varepsilon_2)$$

$$\oplus V(3\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3) \oplus V(3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3)$$

$$= V(2\omega_1 + \omega_2) \oplus V(\omega_1 + 3\omega_2) \oplus V(2\omega_2)$$
$$\oplus V(\omega_1)$$

since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$.

Ex(2) $\mathfrak{g} = \mathfrak{sl}(4, F)$, $\lambda = \omega_1 + \omega_2 = \{2,1\}$, $\mu = 2\omega_2 = \{2^2\}$.

$$V(\lambda) \otimes V(\mu) \cong V(4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3) \oplus V(4\varepsilon_1 + 3\varepsilon_2)$$

$$\oplus V(3\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3) \oplus V(3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3)$$

$$\oplus V(3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4) \oplus V(2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4)$$

$$= V(2\omega_1 + \omega_2 + \omega_3) \oplus V(\omega_1 + 3\omega_2) \oplus V(2\omega_2 + \omega_3)$$

$$\oplus V(\omega_1 + 2\omega_3) \oplus V(\omega_1 + \omega_2) \oplus V(\omega_3)$$

since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0$, $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, $\omega_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Ex(3) $\mathfrak{g} = \mathfrak{sl}(n, F)$, $n \geq 5$, $\lambda = \omega_1 + \omega_2 = \{2,1\}$, $\mu = 2\omega_2 = \{2^2\}$.

$$V(\lambda) \otimes V(\mu) \cong V(2\omega_1 + \omega_2 + \omega_3) \oplus V(\omega_1 + 3\omega_2)$$

$$\oplus V(2\omega_2 + \omega_3) \oplus V(\omega_1 + 2\omega_3)$$

$$\oplus V(\omega_1 + \omega_2 + \omega_4) \oplus V(\omega_3 + \omega_4)$$

since $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$, $\omega_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ $1 \leq i \leq n-1$.

$\oplus V(2\varepsilon_1+2\varepsilon_2+\varepsilon_3) \oplus \cancel{V(3\varepsilon_1+2\varepsilon_2)}$

$\oplus V(\cancel{3\varepsilon_1+\varepsilon_2+\varepsilon_3})$

$= V(5\omega_1+\omega_2) \oplus V(2\omega_1+\omega_2) \oplus 2V(\cancel{\omega_1+3\omega_2})$

$\oplus V(2\omega_2) \oplus \cancel{V(\omega_1)} \oplus \cancel{V(2\omega_1+\omega_2)}$

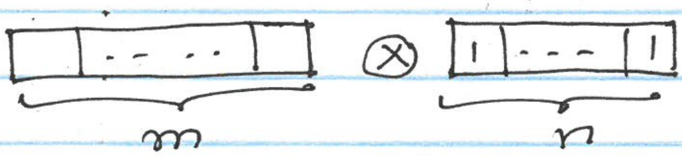
$\cancel{\oplus V(3\omega_1+2\omega_2)}$

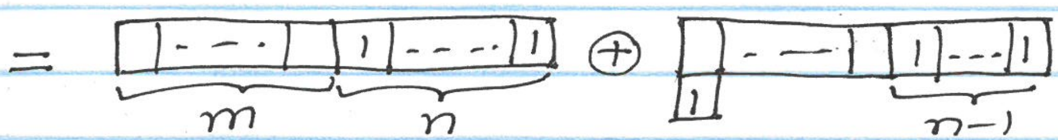
$\cancel{\oplus V(3\omega_2) \oplus V(4\omega_1)}$

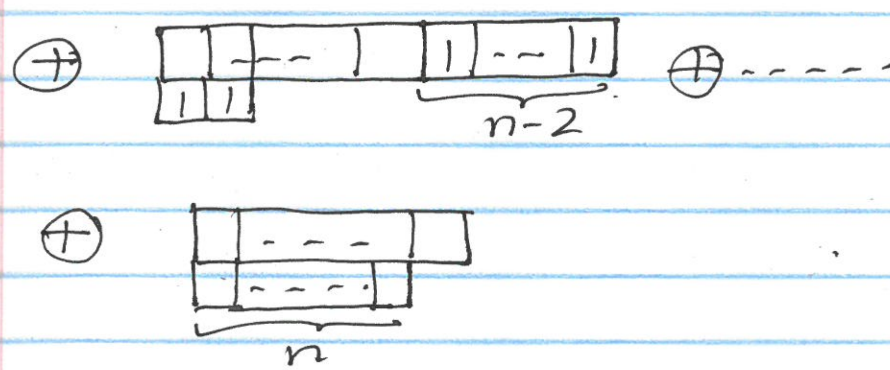
Ex(2) $\mathfrak{g} = sl(2,F)$,

$\lambda = m\omega_1$, $\mu = n\omega_1$, $m \geq n$.

$$V(\lambda) \otimes V(\mu) = V(m) \otimes V(n)$$







~~[illegible]~~

$V(m\omega_1) \otimes V(n\omega_1)$

$= V((m+n)\varepsilon_1) \oplus V((m+n-1)\varepsilon_1 + \varepsilon_2)$

$\oplus V((m+n-2)\varepsilon_1 + 2\varepsilon_2) + \cdots \oplus V(m\varepsilon_1 + n\varepsilon_2)$

$= V(m+n) \oplus V(m+n-2) \oplus V(m+n-4)$

$\oplus - - - - \oplus V(m-n)$

which is the Clebsch-Gordon formula.