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$\mathfrak{g} = sl(n, F)$, $\lambda \in P^+_{n-1}$

$$\Rightarrow \lambda = \sum_{i=1} m_i \omega_i , \; m_i \in \mathbb{Z}_{\geq 0}$$

$V(\lambda)$ fin. dim'l irred. $\mathfrak{g}$-module with highest wt. λ and generated by the highest wt vector v_λ.

$$\dim V(\lambda)_\lambda = 1$$

$\mu \in \mathfrak{h}^*$ is a wt. if $\dim V(\lambda)_\mu \neq 0$

In such case $\mu = \lambda - \sum_{i=1}^{n-1} k_i \alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned}
\lambda \in P^+ \Rightarrow \lambda &= \sum_{i=1}^{n-1} m_i \omega_i \\
&= \sum_{i=1}^{n-1} m_i (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i) \\
&= (m_1 + m_2 + \cdots + m_{n-1}) \varepsilon_1 \\
&\quad + (m_2 + m_3 + \cdots + m_{n-1}) \varepsilon_2 \\
&\quad + \cdots + m_{n-1} \varepsilon_{n-1} \\
&= \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_{n-1} \varepsilon_{n-1}
\end{aligned}$$

where $\lambda_j = \sum_{i=j}^{n-1} m_i$.

$$\Rightarrow \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0$$

Take $m = \sum_{i=1}^{n-1} \lambda_i \in \mathbb{Z}_{>0}$ if $\lambda \neq 0$.

$$m \vdash \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0\}$$

called a partition of m. In such case we say $\lambda \vdash m$.

Choose k to be the smallest integer such that $\lambda_k \neq 0$ & $\lambda_j = 0$ for $j > k$. Then

$$\lambda \vdash \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$$

Conversely, suppose

$\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$ be a partition of $m \in \mathbb{Z}_{>0}$. Set

$$\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_k \varepsilon_k, \quad (k \leq n-1)$$

$$= (\lambda_1 - \lambda_2)\omega_1 + (\lambda_2 - \lambda_3)\omega_2 + \cdots + \lambda_k \omega_k \in P^+.$$

$k = l(\lambda)$ called the length of the partition.

$\Rightarrow$ There is a 1-1 correspondence between the set P^+ and the set of partitions $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$ with $k \leq n-1$.

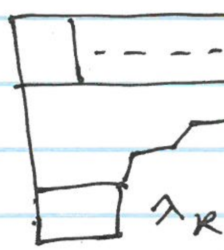
Fix a partition

$$\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\} \text{ of}$$

$$m = \sum_{i=1}^{k} \lambda_i, \quad k \leq n-1.$$

Then $\lambda = \sum_{i=1}^{k} \lambda_i \varepsilon_i \in P^+$, $V(\lambda)$ is the irred. fin. dim'l $\mathfrak{g}$-module.

We associate to this partition λ its Ferrer's diagram (or Young diagram)

$F(\lambda) =$ 

with λ_i left justified boxes in the ith row.

<u>Ex (1)</u> $\lambda = 3\omega_1 + 2\omega_2$

$$= 3\varepsilon_1 + 2\varepsilon_1 + 2\varepsilon_2$$

$$= 5\varepsilon_1 + 2\varepsilon_2 = \{5, 2\}$$

$$F(\lambda) = \begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \cline{1-2}\end{array}$$

$F(\lambda)$ is called the frame of λ.

Standard Tableau:

A standard tableau of shape λ is a filling of the frame $F(\lambda)$ with numbers $\{1, 2, \cdots, m\}$, $m \vdash \lambda$ in such a way that the entries increase from left to right across each row and top to bottom across each column.

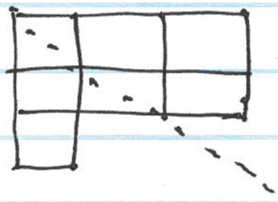
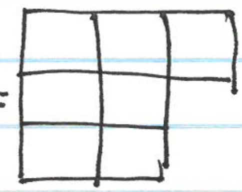
Ex (2) $\lambda = 2\omega_2 + \omega_3 = 3\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3$
$= \{3^2, 1\} \vdash 7 = m$

$$F(\lambda) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \multicolumn{1}{|c|}{7} \\ \cline{1-1}\end{array}$$

The number of standard tableaux of shape λ is given by the well known "hook length formula".

The conjugate λ^* of the partition λ has the frame $F(\lambda^*)$ which is obtained from $F(\lambda)$ by reflecting about the main diagonal.

Ex (3) $\lambda = \{3^2, 1\}$

$F(\lambda) =$  , $F(\lambda^*) =$ 

$\Rightarrow \lambda^* = \{3, 2^2\}$.

Let $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_k > 0\}$ and

$\lambda^* = \{\lambda_1^* \geq \cdots \geq \lambda_k^* > 0\}$

The "hook" of the (i,j)-box in $F(\lambda)$ is given by

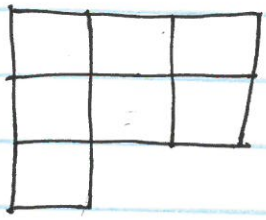
$$h(i,j) = \underbrace{(\lambda_i - i)}_{\substack{\#\text{ of boxes} \\ \text{in the } i\text{th} \\ \text{row of } F(\lambda) \\ \text{to the right} \\ \text{of the } (i,j)\text{-box}}} + \underbrace{(\lambda_j^* - j)}_{\substack{\#\text{ of boxes} \\ \text{in the } j\text{th} \\ \text{column of} \\ F(\lambda) \text{ below} \\ \text{the } (i,j)\text{-box}}} + \underset{(i,j)\text{-box}}{1}$$

$$h(\lambda) = \prod_{(i,j)} h(i,j)$$

is the hook length of λ.

Thm: The # of standard tableaux of shape λ with $\lambda \vdash m$ is equal to $\dfrac{m!}{h(\lambda)}$.

Ex (4) $\lambda = \{3^2, 1\} \vdash 7$

$F(\lambda) =$  , $\lambda^* = \{3, 2^2\}$

$h(\lambda) = (5)(3)(2)(4)(2)(1)(1)$

# of standard tableaux of shape λ

$$= \frac{m!}{h(\lambda)} = \frac{7!}{(5)(4)(3)(2)(2)} = 21$$

Consider $V = F^n$ is an irred. n-dim'l $\mathfrak{g} = sl(n, F)$-module under matrix multiplication.

Recall $V = V(\overset{\lambda}{\overset{\shortparallel}{\omega_1}}) = V(\varepsilon_1)$ with highest wt. vector $v_\lambda = (1, 0, \ldots, 0)^T$.

For a positive integer m, the m-fold tensor product

$$\bigotimes^m V$$

is completely reducible by the Weyl's Thm.

<u>Thm:</u>

$$\bigotimes^m V \cong \bigoplus_{\substack{\lambda \vdash m \\ \ell(\lambda) \leq n}} \frac{m!}{h(\lambda)} V(\lambda)$$

where $V(\lambda)$ is the irred. $\mathfrak{sl}(n, F)$ module with highest weight

$$\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_n \varepsilon_n \in P^+$$

$(\lambda \vdash \{\lambda_1 \geq \cdots \geq \lambda_n \geq 0\})$.

<u>Ex (1)</u> $\mathfrak{g} = \mathfrak{sl}(2, F)$, $V = F^2 = V(\varepsilon_1) = V(1)$

$(\varepsilon_1 + \varepsilon_2 = 0)$.

$m = 2$: $\bigotimes^2 V = V \otimes V$

$2 \vdash 2$ 

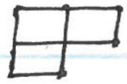
$2 \vdash 1+1$ 

$$\bigotimes^2 V = \frac{2!}{(2)(1)} V(2\varepsilon_1) \oplus \frac{2!}{(2)(1)} V(\varepsilon_1 + \varepsilon_2)$$

$$= V(2) \oplus V(0)$$

$m = 3$: $\bigotimes^3 V = V \otimes V \otimes V$

$3 \vdash \{3\}$: 

$3 \vdash \{2,1\}$ 

$3 \vdash \{1,1,1\}$ X since length > 2.

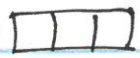
$$\bigotimes^3 V = \frac{3!}{(3)(2)(1)} V(3\varepsilon_1) \oplus \frac{3!}{(3)(1)(1)} V(2\varepsilon_1 + \varepsilon_2)$$

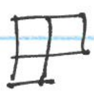
$$= V(3) \oplus 2V(1)$$

Ex (2) $\mathfrak{g} = \mathfrak{sl}(3, F)$, $V = F^3$

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$

$$\bigotimes^3 V = V \otimes V \otimes V$$

$3 \vdash 3$: 

$3 \vdash \{2,1\}$: 

$3 \vdash \{1,1,1\}$: 

$$= \frac{3!}{(3)(2)(1)} V(3\varepsilon_1) \oplus \frac{3!}{(3)(1)(1)} V(2\varepsilon_1 + \varepsilon_2) \oplus \frac{3!}{(3)(2)(1)} V(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$= V(3\omega_1) \oplus 2V(\omega_1 + \omega_2) \oplus V(0)$$