Thm: (1) Every fin. dim'l highest weight module is irreducible

(2) In any fin-dim'l highest weight module, there is a unique highest weight and a unique highest weight vector up to a scalar multiple.

(3) If two fin-dim'l highest weight modules have the same highest weight then they are isom.

Pf (1) Let V be a $\mathfrak{g}$-module gen. by the highest wt. vector v_λ with highest weight λ.

By Weyl's Thm. V is completely reducible. Suppose

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$$

U_j are irred. submodules. Then

$$v_\lambda = u_1 + u_2 + \cdots + u_k, \quad u_j \in U_j$$

$$\lambda(h) v_\lambda = h.v_\lambda = h.(u_1 + u_2 + \cdots + u_k)$$

$$= h.u_1 + h.u_2 + \cdots + h.u_k$$

$$\lambda(h)u_1 + \lambda(h)u_2 + \cdots + \lambda(h)u_k$$

$$\Rightarrow h.u_j = \lambda(h)u_j \quad \forall\, h \in \mathfrak{h}.$$

$\Rightarrow u_1, u_2, \cdots, u_k$ are weight vectors with weight λ.

Since $\dim V_\lambda = 1$, we have a contradiction.

$\Longrightarrow V$ irreducible.

(2) ~~Let V be another g-module~~
Suppose V has another highest weight μ. Then

$$\mu = \lambda - \sum_{i=1}^{n-1} m_i \alpha_i, \quad m_i \geq 0.$$

$$\Rightarrow \lambda - \mu = \sum_{i=1}^{n-1} m_i \alpha_i \Rightarrow ht(\lambda - \mu) > 0$$

Let w be a highest weight vector with highest wt. μ. Then w generates a proper submodule W of V. By (1), V is irreducible which is a contradiction. Hence $\mu = \lambda$. Now the highest weight vector v_λ with highest wt. λ is unique up to scalar multiple since $\dim V_\lambda = 1$.

(3) Let V' be another highest weight module with highest weight λ. Consider the $\mathfrak{g}$-module

$$V \oplus V'.$$

Let v_λ, u_λ be the highest wt. vectors, with highest wt. λ in V and V' respectively. Then

$$w = v_\lambda + u_\lambda \in V \oplus V'$$

$\forall\, h \in \mathfrak{h}$,
$$\begin{aligned} h.w &= h.(v_\lambda + u_\lambda) \\ &= h.v_\lambda + h.u_\lambda \\ &= \lambda(h) v_\lambda + \lambda(h) u_\lambda \\ &= \lambda(h)(v_\lambda + u_\lambda) = \lambda(h) w \end{aligned}$$

$\Rightarrow$ w has weight λ. Since

$$\mathfrak{n}^+ . v_\lambda = 0 = \mathfrak{n}^+ . u_\lambda, \text{ we have}$$

$$\mathfrak{n}^+ . w = 0$$

$\Rightarrow$ w is a highest wt. vector with highest weight λ. Let W be the submodule of $V \oplus V'$ gen. by w.

By (1), V, V', W are irreducible.

Consider the projection homomorphism

$$\pi : V \oplus V' \longrightarrow V$$

$$\pi|_W : W \longrightarrow V \text{ hom.}$$

with $\pi(w) = v_\lambda$. Since W and V are irred.

$$W \cong V,$$

Similarly $W \cong V' \Longrightarrow V \cong V'$. //

<u>Defn:</u> Recall $\Pi^\vee = \{ h_i = E_{ii} - E_{i+1,i+1} \mid 1 \le i \le n-1 \}$

and $\mathfrak{h} = \operatorname{span}\{ h_i \mid 1 \le i \le n-1 \}$.
A weight $\lambda \in \mathfrak{h}^*$ is integral
if $\lambda(h_i) \in \mathbb{Z}$, $1 \le i \le n-1$.
If $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$, $1 \le i \le n-1$, then
we say λ is a dominant integral weight.

Recall the weight lattice

$$P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \cdots \oplus \mathbb{Z}\omega_{n-1}$$

where $\omega_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ &

$$\omega_i(h_j) = \delta_{ij}$$

The set $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0},\ 1 \leq i \leq n-1\}$ is called the set of dominant integral weights.

<u>Thm:</u> Let V be an irreducible finite dim'l $\mathfrak{g}$-module with field F alg. closed & char$(F)=0$. Then V has a height weight vector v_λ with the dominant integral highest weight λ.

<u>Pf:</u> By a previous result V has a highest weight vector v_λ with highest weight λ. For each $1 \leq i \leq n-1$,

$$\text{span}\{h_i, e_i, f_i\} \cong sl(2, F)$$

Then v_λ generates an irred. $sl(2,F)$-module. By $sl(2,F)$

representation theory $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$.

$\Rightarrow$ $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n-1$

$\Rightarrow$ λ is dominant integral.

Recall $\lambda \in P^+$, $V(\lambda)$ be a finite dimension highest weight module with highest vector v_λ of weight λ $\Rightarrow$ $V(\lambda)$ irred.

$\Rightarrow$ The set of irred. finite dim'l $\mathfrak{g}$-modules is in 1-1 correspondence with P^+. (We are assuming F alg. closed & char $F = 0$, ex. $F = \mathbb{C}$)

Let $\mathfrak{g} = sl(n, \mathbb{C}) = \langle e_i, f_i, h_i \mid 1 \le i \le n-1 \rangle$

Let $\lambda \in P^+$ and $V(\lambda)$ be the irred. ~~highest~~ fin. dim'l $\mathfrak{g}$-module gen. by v_λ with highest weight λ.

$$\lambda \in P^+ \Rightarrow \lambda = \sum_{i=1}^{n-1} k_i \omega_i, \quad k_i \in \mathbb{Z}_{\geq 0}$$

$$= \sum_{i=1}^{n-1} k_i (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i)$$

$$= (k_1 + k_2 + \cdots + k_{n-1})\varepsilon_1 + (k_2 + k_3 + \cdots + k_{n-1})\varepsilon_2 + \cdots + (k_{n-2} + k_{n-1})\varepsilon_{n-2} + k_{n-1}\varepsilon_{n-1}$$

$$= \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_{n-1}\varepsilon_{n-1}$$

where $\lambda_i = \sum_{j=i}^{n-1} k_j \in \mathbb{Z}_{\geq 0}$

Then $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1}$

Denote $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}\}$

Let $m = \sum_{i=1}^{n-1} \lambda_i$ and choose k such that $\lambda_k > 0$, $\lambda_j = 0$ for $j > k$.

Then $m = \sum_{i=1}^{k} \lambda_i$

$\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$ is called a partition of m.

Conversely, suppose we have a partition $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$ of $m = \sum_{i=1}^{k} \lambda_i$. Then

$$\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_k \varepsilon_k$$
$$= (\lambda_1 - \lambda_2)\omega_1 + (\lambda_2 - \lambda_3)\omega_2 + \cdots + \lambda_k \omega_k$$
$$\in P^+$$

$\Rightarrow P^+$ is in 1-1 correspondence with set of partitions
$$\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0\}.$$