$\mathfrak{g} = \mathfrak{sl}(n, F)$, char $F = 0$.

Δ = set of roots
$= \{ \pm(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}) \mid 1 \leq i < j \leq n \}$
$= \Delta_+ \sqcup \Delta_-$

$\lambda \in P = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_{n-1}$
$\alpha \in \Delta$

V $\mathfrak{g}$-module

$$V_\lambda = \{ v \in V \mid h.v = \lambda(h)v \ \forall h \in \mathfrak{h} \}$$

Assume $V_\lambda \neq 0 \Rightarrow \lambda$ is a weight.

Let $x \in \mathfrak{g}_\alpha$, $v \in V_\lambda$, $h \in \mathfrak{h}$

$x.v \in V$

$$h.(x.v) = x.(h.v) + [h, x].v$$
$$= \lambda(h)(x.v) + \alpha(h)(x.v)$$
$$= (\lambda + \alpha)(h)(x.v) \quad \forall h \in \mathfrak{h}$$

$$\Rightarrow \mathfrak{g}_\alpha . V_\lambda \subseteq V_{\lambda+\alpha}.$$

Highest weight vector:

$v \in V$ is a highest weight vector with highest weight λ if

(1) $v \in V_\lambda$, and

(2) $\mathfrak{n}^+ . v = 0$

($\Rightarrow \forall\, x \in \mathfrak{g}_\alpha,\ \alpha \in \Delta_+$, $x . v = 0$)

<u>Lemma</u>: Suppose F algebraically closed. Then any finite dimensional $\mathfrak{g}$-module V has a highest weight vector.

<u>Pf</u>: Recall the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ is solvable. V is a $\mathfrak{b}$-module by restriction. By Lie's Theorem $\mathfrak{b}$ has a common eigenvector $v \in V$. That means:

$\forall$ ~~[illegible]~~ $x \in \mathfrak{b}$, $x . v = \lambda(x) v$, $\lambda \in \mathfrak{b}^*$

$\Rightarrow \exists\, \lambda \in \mathfrak{h}^*$ s.t. $h . v = \lambda(h) v \ \forall\, h \in \mathfrak{h}$

$\Rightarrow v \in V_\lambda$

Furthermore, since $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}^+$

$\forall\, x \in \mathfrak{n}^+,\ x = \sum_i [y_i, y_i'],\ y_i, y_i' \in \mathfrak{b}$

$$\begin{aligned} \Rightarrow x \cdot v &= \sum_i [y_i, y_i'] \cdot v \\ &= \sum_i \left(y_i \cdot (y_i' \cdot v) - y_i' \cdot (y_i \cdot v) \right) \\ &= \sum_i \left(\lambda(y_i) \lambda(y_i') v - \lambda(y_i') \lambda(y_i) v \right) \\ &= 0 \end{aligned}$$

$\Rightarrow$ v is a highest weight vector with highest weight λ. //

Suppose V is a finite dimensional irreducible $\mathfrak{g}$-module.

$\Rightarrow$ V contains a highest weight vector v_λ with highest weight $\lambda \in \mathfrak{h}^*$.

Let W be the submodule generated by v. This means:

Since $\mathfrak{b} \cdot v = \mathfrak{h} \cdot v \in V_\lambda$ and

$[e_i, f_j] = \delta_{ij} h_i$

$\forall\, w \in W$,

$$w = f_{i_1} f_{i_2} \cdots f_{i_k} \cdot v_\lambda,\ \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n-1\}$$

Recall $f_{i_j} \in \mathfrak{g}_{-\alpha_{i_j}}$, $\alpha_{i_j} \in \Pi$.

$$\Rightarrow w \in V_{\lambda - \alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_k}} = V_\mu$$

where $\mu = \lambda - \sum_{i=1}^{n-1} m_i \alpha_i \in \mathfrak{h}^*$, $m_i \geq 0$.

Since V is irreducible, $W = V = V(\lambda)$.

Recall $\forall\, \alpha = \sum_{i=1}^{n-1} k_i \alpha_i$, $ht(\alpha) = \sum_{i=1}^{n-1} k_i$.

Define $Q^+ = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \mathbb{Z}_{\geq 0}\alpha_2 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_{n-1}$

called the positive root lattice.

<u>Defn</u>: For $\mu, \nu \in \mathfrak{h}^*$, we say $\mu \geq \nu$ if $\mu - \nu \in Q^+$

$$V(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} V(\lambda)_\mu$$

- $V(\lambda)_\mu \neq 0 \Leftrightarrow \mu = \lambda - \sum m_i \alpha_i, \ m_i \geq 0.$

$$\Rightarrow \lambda - \mu = \sum_{i=1}^{n-1} m_i \alpha_i \in Q^+$$

$$\Rightarrow \mu \leq \lambda$$

$$\Rightarrow \dim V(\lambda)_\lambda = 1$$

$\Rightarrow$ the highest weight vector v_λ is unique up to scalar multiple.

<u>Thm:</u> (1) Every finite dimensional highest weight $\mathfrak{g}$-module is irreducible.

(2) In any finite dimensional highest weight $\mathfrak{g}$-module there is a unique highest and ~~the~~ a unique highest weight vector up to scalar multiple.

(3) If ~~two~~ fin. dim'l highest weight $\mathfrak{g}$-modules have the same highest weight, then they are isomorphic.

(<u>Note</u>: A $\mathfrak{g}$-module V is a highest weight module if it contains a highest weight vector v and v generates the module V.

(i.e. $u \in V$, then $u = x_{i_1} \cdots x_{i_k} v$, $x_{i_j} \in \mathfrak{g}$))